Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 57 (2007) 1569–1600
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Spacelike parallels and evolutes in Minkowski pseudo-spheres

Shyuichi Izumiya*, Masatomo Takahashi

Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan

Received 10 June 2006; accepted 25 January 2007
Available online 8 February 2007

Abstract

We consider extrinsic differential geometry on spacelike hypersurfaces in Minkowski pseudo-spheres (hyperbolic space, de Sitter space and the lightcone). In the previous paper [S. Izumiya, Legendrian dualities and spacelike hypersurfaces in the lightcone, Preprint] we have shown a basic Legendrian duality theorem between pseudo-spheres. We define the spacelike parallels by using the basic Legendrian duality theorem. This definition unifies the notions of parallels of spacelike hypersurfaces in pseudo-spheres. We also define the evolute as the locus of singularities of the spacelike parallels. These notions are investigated as applications of Lagrangian or Legendrian singularity theory. We consider geometric properties of non-singular spacelike hypersurfaces corresponding to singularities of spacelike parallels or evolutes.

MSC: 53A35; 57R45; 58Kxx

Keywords: Spacelike parallels; Evolutes; Caustics; Lagrangian singularities; Legendrian singularities

1. Introduction

In this paper we describe some results of the project constructing the extrinsic differential geometry on submanifolds of Minkowski pseudo-spheres (cf. [12–18]). In [18] we have shown a basic Legendrian duality theorem between pseudo-spheres in Minkowski space in order to develop an extrinsic differential geometry for spacelike hypersurfaces in pseudo-spheres. Especially, we have stuck to spacelike hypersurfaces in the lightcone motivated by the results of Asperti and Dajczer [3] on conformally flat Riemannian manifolds. For a spacelike hypersurface in the lightcone, we cannot define the normal vector because the metric is degenerate. However, we have defined the lightlike Gauss image of a spacelike hypersurface in the lightcone as a direct application of the basic duality theorem. The derivative of the lightcone Gauss image can be interpreted as a linear transformation on the tangent space of the spacelike hypersurface which is called the lightcone Weingarten map. Therefore we have the lightlike principal curvatures as the eigenvalues of the lightcone Weingarten map. It follows that we have the lightcone Gauss–Kronecker curvature of the hypersurface as the product of the lightlike principal curvatures. We can also apply the Legendrian duality theorem to spacelike hypersurfaces in hyperbolic space or de Sitter space. For hypersurfaces in hyperbolic

* Corresponding author.
E-mail addresses: izumiya@math.sci.hokudai.ac.jp (S. Izumiya), takahashi@math.sci.hokudai.ac.jp (M. Takahashi).

0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2007.01.008

space, we have reconstructed the hyperbolic Gauss–Kronecker curvature in [18] by using the basic Legendrian duality theorem which was originally introduced in [12].

On the other hand, the notions of parallels and evolutes (focal sets) play important roles in the classical differential geometry for hypersurfaces in Euclidean space. Basic properties of such singular hypersurfaces were investigated by many people [2,4,24]. As a consequence, we can interpret that these results on evolutes describe the contact of hypersurfaces with hyperspheres (i.e., totally umbilic hypersurfaces with non-zero Gauss curvatures). It is called the "spherical (or round) geometry" of hypersurfaces in Euclidean space.

In [13,14] we have studied the evolutes of hypersurfaces in hyperbolic space and discovered some examples of hypersurfaces that the evolutes are spilt out of hyperbolic space. Some parts of the evolutes of such examples are located in de Sitter space. Therefore we have defined the notion of hyperbolic evolutes and de Sitter evolutes of hypersurfaces in hyperbolic space. In the Euclidean space, it has been known that the evolute of a hypersurface is the locus of singularities of the parallels of the original hypersurfaces. This means that the corresponding notion of the parallels for hypersurfaces in hyperbolic space might be also spilt out of hyperbolic space. Under such observation, we introduce the notion of spacelike parallels and evolutes in Minkowski pseudo-spheres. Here, Minkowski pseudo-spheres are hyperbolic space, de Sitter space or the lightcone (cf. Section 2). In Section 12 of [18] we remarked that the corresponding notion of parallels and evolutes of spacelike hypersurfaces in the lightcone have quite different properties from the parallels and the evolutes in Euclidean space. Especially if we consider a spacelike hypersurface in the lightcone, the parallels of the spacelike hypersurfaces are never located in the lightcone. Of course the evolute of the hypersurface are also in the same situation. This fact is quite different from the other hypersurfaces theories. Minkowski space is originally from the relativity theory in Physics (i.e., Lorentzian geometry in Mathematics). We refer to the book [23] for general properties of Minkowski space and Lorentzian geometry.

In Section 2 we give a brief review on the previous results on spacelike hypersurfaces in Minkowski pseudo-spheres. Especially, the basic Legendrian duality theorem in [18] is stated. We also review classification results on totally umbilic spacelike hypersurfaces in pseudo-spheres. We consider such totally umbilic spacelike hypersurfaces as "model hypersurfaces". Spacelike parallels and caustics are defined in Section 3 as an application of the basic Legendrian duality theorem. By definition we can show that the caustics is the locus of singularities of spacelike parallels. According to the classification results of the totally umbilic spacelike hypersurfaces in pseudo-spheres, the evolutes is defined in Section 4. In order to study parallels and evolutes, we introduce timelike height functions and spacelike height functions in Section 5. By the direct calculation, we can show that the above notions of caustics and evolutes are the same. In Sections 6 and 7 we study parallels and caustics from the viewpoint of Lagrangian or Legendrian singularity theory. In Section 8 we study the geometric meaning of both the singularities of parallels and evolute from the viewpoint of the contact with families of model hypersurfaces (totally umbilic hypersurfaces). We study generic properties in Section 9. In Section 10 we apply the classification results in [10,28] to the case for $n = 3$ and draw some pictures.

We shall assume throughout the whole paper that all the maps and manifolds are C^∞ unless the contrary is explicitly stated.

2. Basic concepts and notations

In this section we prepare basic notions on Minkowski space and contact geometry. Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \ldots, x_n) | x_i \in \mathbb{R},\ i = 0, 1, \ldots, n\}$ be an $(n+1)$-dimensional vector space. For any vectors $\boldsymbol{x} = (x_0, \ldots, x_n)$, $\boldsymbol{y} = (y_0, \ldots, y_n)$ in $\mathbb{R}^{n+1}$, the *pseudo-scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i$. The space $(\mathbb{R}^{n+1}, \langle , \rangle)$ is called *Minkowski* $(n+1)$*-space* and denoted by $\mathbb{R}^{n+1}_1$.

We say that a vector $\boldsymbol{x}$ in $\mathbb{R}^{n+1} \setminus \{\boldsymbol{0}\}$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, = 0$ or < 0 respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^{n+1}$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. Given a vector $\boldsymbol{n} \in \mathbb{R}^{n+1}_1$ and a real number c, the hyperplane with pseudo-normal $\boldsymbol{n}$ is given by

$$HP(\boldsymbol{n}, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{n} \rangle = c\}.$$

We say that $HP(\boldsymbol{n}, c)$ is a *spacelike*, *timelike* or *lightlike hyperplane* if $\boldsymbol{n}$ is timelike, spacelike or lightlike respectively.

We have the following three kinds of pseudo-spheres in $\mathbb{R}^{n+1}_1$: the *hyperbolic n-space* is defined by

$$H^n(-1) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\},$$

the *de Sitter n-space* by

$$S_1^n = \{\boldsymbol{x} \in \mathbb{R}_1^{n+1} | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}$$

and the (*open*) *lightcone* by

$$LC^* = \{\boldsymbol{x} \in \mathbb{R}_1^{n+1} \setminus \{\boldsymbol{0}\} | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}.$$

We now review some properties of contact manifolds and Legendrian submanifolds. Let N be a $(2n+1)$-dimensional smooth manifold and K be a tangent hyperplane field on N. Locally such a field is defined as the field of zeros of a 1-form α. The tangent hyperplane field K is *non-degenerate* if $\alpha \wedge (\mathrm{d}\alpha)^n \neq 0$ at any point of N. We say that (N, K) is a *contact manifold* if K is a non-degenerate hyperplane field. In this case K is called a *contact structure* and α is a *contact form*. Let $\phi : N \longrightarrow N'$ be a diffeomorphism between contact manifolds (N, K) and (N', K'). We say that ϕ is a *contact diffeomorphism* if $\mathrm{d}\phi(K) = K'$. Two contact manifolds (N, K) and (N', K') are *contact diffeomorphic* if there exists a contact diffeomorphism $\phi : N \longrightarrow N'$. A submanifold $i : L \subset N$ of a contact manifold (N, K) is said to be *Legendrian* if $\dim L = n$ and $di_x(T_xL) \subset K_{i(x)}$ at any $x \in L$. We say that a smooth fiber bundle $\pi : E \longrightarrow M$ is called a *Legendrian fibration* if its total space E is furnished with a contact structure and its fibers are Legendrian submanifolds. Let $\pi : E \longrightarrow M$ be a Legendrian fibration. For a Legendrian submanifold $i : L \subset E$, $\pi \circ i : L \longrightarrow M$ is called a *Legendrian map*. The image of the Legendrian map $\pi \circ i$ is called a *wavefront set* of i which is denoted by $W(L)$. For any $p \in E$, it is known that there is a local coordinate system $(x_1, \ldots, x_m, p_1, \ldots, p_m, z)$ around p such that

$$\pi(x_1, \ldots, x_m, p_1, \ldots, p_m, z) = (x_1, \ldots, x_m, z)$$

and the contact structure is given by the 1-form

$$\alpha = \mathrm{d}z - \sum_{i=1}^{m} p_i \mathrm{d}x_i$$

(cf. [1], 20.3).

In [18] we have shown the basic duality theorem which is the fundamental tool for the study of spacelike hypersurfaces in Minkowski pseudo-spheres. We now consider the following four double fibrations:

(1) (a) $H^n(-1) \times S_1^n \supset \Delta_1 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0\}$,
(b) $\pi_{11} : \Delta_1 \longrightarrow H^n(-1)$, $\pi_{12} : \Delta_1 \longrightarrow S_1^n$,
(c) $\theta_{11} = \langle \mathrm{d}\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_1$, $\theta_{12} = \langle \boldsymbol{v}, \mathrm{d}\boldsymbol{w} \rangle | \Delta_1$.

(2) (a) $H^n(-1) \times LC^* \supset \Delta_2 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = -1\}$,
(b) $\pi_{21} : \Delta_2 \longrightarrow H^n(-1)$, $\pi_{22} : \Delta_2 \longrightarrow LC^*$,
(c) $\theta_{21} = \langle \mathrm{d}\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_2$, $\theta_{22} = \langle \boldsymbol{v}, \mathrm{d}\boldsymbol{w} \rangle | \Delta_2$.

(3) (a) $LC^* \times S_1^n \supset \Delta_3 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 1\}$,
(b) $\pi_{31} : \Delta_3 \longrightarrow LC^*$, $\pi_{32} : \Delta_3 \longrightarrow S_1^n$,
(c) $\theta_{31} = \langle \mathrm{d}\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_3$, $\theta_{32} = \langle \boldsymbol{v}, \mathrm{d}\boldsymbol{w} \rangle | \Delta_3$.

(4) (a) $LC^* \times LC^* \supset \Delta_4 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = -2\}$,
(b) $\pi_{41} : \Delta_4 \longrightarrow LC^*$, $\pi_{42} : \Delta_4 \longrightarrow LC^*$,
(c) $\theta_{41} = \langle \mathrm{d}\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_4$, $\theta_{42} = \langle \boldsymbol{v}, \mathrm{d}\boldsymbol{w} \rangle | \Delta_4$.

Here, $\pi_{i1}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}$, $\pi_{i2}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{w}$, $\langle \mathrm{d}\boldsymbol{v}, \boldsymbol{w} \rangle = -w_0 \mathrm{d}v_0 + \sum_{i=1}^n w_i \mathrm{d}v_i$ and $\langle \boldsymbol{v}, \mathrm{d}\boldsymbol{w} \rangle = -v_0 \mathrm{d}w_0 + \sum_{i=1}^n v_i \mathrm{d}w_i$.

We remark that $\theta_{i1}^{-1}(0)$ and $\theta_{i2}^{-1}(0)$ define the same tangent hyperplane field over Δ_i which is denoted by K_i. The basic duality theorem is the following theorem:

Theorem 2.1. *Under the same notations as the previous paragraph, each* (Δ_i, K_i) $(i = 1, 2, 3, 4)$ *is a contact manifold and both of* π_{ij} $(j = 1, 2)$ *are Legendrian fibrations. Moreover those contact manifolds are contact diffeomorphic with each other.*

We do not give the proof of the theorem here. However we need the canonical contact diffeomorphism between Δ_1 and Δ_4. We define a smooth mapping

$$\Phi_{14} : \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1 \longrightarrow \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1$$

by $\Phi_{14}(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v} + \boldsymbol{w}, \boldsymbol{v} - \boldsymbol{w})$. The converse mapping is given by

$$\Phi_{41}(\boldsymbol{v}, \boldsymbol{w}) = \left(\frac{\boldsymbol{v} + \boldsymbol{w}}{2}, \frac{\boldsymbol{v} - \boldsymbol{w}}{2} \right).$$

We can also check that $\Phi_{14}(\Delta_1) = \Delta_4$ and $\Phi_{41}(\Delta_4) = \Delta_1$, so that $\Phi_{14}|\Delta_1$ and $\Phi_{41}|\Delta_4$ are diffeomorphism. We can easily check that Φ_{14} and Φ_{41} are contact diffeomorphisms.

We now consider differential geometry of hypersurfaces in pseudo-spheres as applications of the basic theorem. Let

$$\mathcal{L}_1 : U \longrightarrow \Delta_1$$

be a Legendrian embedding and denote that $\mathcal{L}_1(u) = (\boldsymbol{x}^h(u), \boldsymbol{x}^d(u))$. By using the above contact diffeomorphism, we have a Legendrian embedding

$$\mathcal{L}_4 : U \longrightarrow \Delta_4$$

defined by $\mathcal{L}_4(u) = \Phi_{14} \circ \mathcal{L}_1(u)$. We denote that $\mathcal{L}_4(u) = (\boldsymbol{x}^\ell_+(u), \boldsymbol{x}^\ell_-(u))$, so that we have the following relations:

$$\boldsymbol{x}^h(u) = \frac{\boldsymbol{x}^\ell_+(u) + \boldsymbol{x}^\ell_-(u)}{2}, \qquad \boldsymbol{x}^d(u) = \frac{\boldsymbol{x}^\ell_+(u) - \boldsymbol{x}^\ell_-(u)}{2}.$$

We now distinguish three cases as follows:

Case (1) We assume that $\boldsymbol{x}^h : U \longrightarrow H^n(-1)$ is an embedding. In this case $\boldsymbol{x}^\ell_\pm$ are the hyperbolic Gauss indicatrices of $\boldsymbol{x}^h$ which are defined in [12]. Nevertheless, we call these the *lightcone Gauss image* here. We also call $\boldsymbol{x}^d$ the *de Sitter Gauss image* of $\boldsymbol{x}^h$. In [12] we showed that the derivatives of $\boldsymbol{x}^\ell_\pm$ and $\boldsymbol{x}^d$ at u_0 can be considered as linear transformations on the tangent space of $M = \boldsymbol{x}^h(U)$ at $p = \boldsymbol{x}^h(u_0)$. We respectively call $S^\ell_\pm(p) = -\mathrm{d}\boldsymbol{x}^\ell_\pm(u_0)$ and $S^d(p) = -\mathrm{d}\boldsymbol{x}^d(u_0)$ the *lightcone shape operator* and the *de Sitter shape operator* of $M = \boldsymbol{x}^h(U)$ at $p = \boldsymbol{x}^h(u_0)$. We denote the eigenvalue of $S^\ell_\pm(p)$ by $\kappa^\ell_\pm(p)$ and the eigenvalue of $S^d(p)$ by $\kappa^d(p)$. By the relation $\boldsymbol{x}^h \pm \boldsymbol{x}^d = \boldsymbol{x}^\ell_\pm$, we have a relation $S^\ell_\pm(p) = -id_{T_pM} \pm S^d(p)$ under the identification of U and M through $\boldsymbol{x}^h$. Therefore, $S^\ell_\pm(p)$ and $S^d(p)$ have the same eigenvectors and we have a relation that $\kappa^\ell_\pm(p) = -1 \pm \kappa^d(p)$.

We now define the notion of Gauss–Kronecker curvatures of $M = \boldsymbol{x}^h(U)$ at $p = \boldsymbol{x}^h(u_0)$ as follows:

$K^\pm_\ell(u_0) = \det S^\ell_\pm(p)$; The lightcone Gauss–Kronecker curvature,

$K_d(u_0) = \det S^d(p)$; The de Sitter Gauss–Kronecker curvature.

We remark that $K^\pm_\ell(u)$ is called the hyperbolic Gauss–Kronecker curvature in [12]. We say that a point $u_0 \in U$ or $p = \boldsymbol{x}^h(u_0)$ is an *umbilic point* if $S^\ell_\pm(p) = \kappa^\ell_\pm(p) id_{T_pM}$. Since the eigenvectors of $S^\ell_\pm(p)$ and $S^d(p)$ are the same, the above condition is equivalent to the condition $S^d(p) = \kappa^d(p) id_{T_pM}$. We say that $M = \boldsymbol{x}^h(U)$ is *totally umbilic* if all points on M are umbilic. Here, we consider the following model hypersurfaces in hyperbolic space. We consider the intersection of $H^n(-1)$ with a hyperplane in $\mathbb{R}^{n+1}_1$:

$$HH(\boldsymbol{n}, c) = HP(\boldsymbol{n}, c) \cap H^n(-1).$$

We say that $HH(\boldsymbol{n}, c)$ is a *hypersphere* if $\boldsymbol{n}$ is timelike, a *equidistant hypersurface* if $\boldsymbol{n}$ is spacelike and a *hyperhorosphere* if $\boldsymbol{n}$ is lightlike. Especially the equidistant hypersurface $HH(\boldsymbol{n}, 0)$ is called a *hyperplane*. In [12] it has been shown the following proposition.

Proposition 2.2. *Suppose that $M = \boldsymbol{x}^h(U) \subset H^n(-1)$ is totally umbilic. Then $\kappa^\ell_\pm(p)$ is constant $\kappa^\ell_\pm$. Under this condition, we have the following classification:*

(1) *Suppose that $(\kappa^\ell_\pm)^2 + 2\kappa^\ell_\pm \neq 0$.*

(a) *If* $(\kappa_\pm^\ell)^2 + 2\kappa_\pm^\ell > 0$*, then* M *is a part of hypersphere*

$$HH\left(\boldsymbol{c}, \frac{-\kappa_\pm^\ell - 1}{\sqrt{(\kappa_\pm^\ell)^2 + 2\kappa_\pm^\ell}}\right),$$

where

$$\boldsymbol{c} = \frac{1}{\sqrt{(\kappa_\pm^\ell)^2 + 2\kappa_\pm^\ell}}(\kappa_\pm^\ell \boldsymbol{x}^h(u) + \boldsymbol{x}_\pm^\ell(u)) \in H^n(-1)$$

is a constant timelike vector.

(b) *If* $(\kappa_\pm^\ell)^2 + 2\kappa_\pm^\ell < 0$*, then* M *is a part of an equidistant hypersurface*

$$HH\left(\boldsymbol{c}, \frac{-\kappa_\pm^\ell - 1}{\sqrt{-(\kappa_\pm^\ell)^2 - 2\kappa_\pm^\ell}}\right),$$

where

$$\boldsymbol{c} = \frac{1}{\sqrt{-(\kappa_\pm^\ell)^2 - 2\kappa_\pm^\ell}}(\kappa_\pm^\ell \boldsymbol{x}^h(u) + \boldsymbol{x}_\pm^\ell(u)) \in S_1^n$$

is a constant spacelike vector. In particular, if $\kappa_\pm^\ell = -1$*, then* M *is a part of hyperplane* $HH(\boldsymbol{c}, 0)$*, where* $\boldsymbol{c} = \boldsymbol{x}^d(u)$ *is a constant spacelike vector.*

(2) *If* $(\kappa_\pm^\ell)^2 + 2\kappa_\pm^\ell = 0$*, then* M *is a part of a hyperhorosphere* $HH(\boldsymbol{c}, -\kappa_\pm^\ell - 1)$*, where* $\boldsymbol{c} = \kappa_\pm^\ell \boldsymbol{x}^h(u) + \boldsymbol{x}_\pm^\ell(u)$ *is a constant lightlike vector.*

Case (2) We assume that $\boldsymbol{x}^d : U \longrightarrow S_1^n$ is an embedding. Since $\mathcal{L}_1$ is a Legendrian embedding, $\boldsymbol{x}^d$ is a spacelike embedding. (i.e., an embedding and $\boldsymbol{x}_{u_i}^d$, $(i = 1, \ldots, n-1)$ are spacelike vectors). We also call $\boldsymbol{x}_\pm^\ell$ the *lightcone Gauss image* and $\boldsymbol{x}^h$ the *hyperbolic Gauss image* of $\boldsymbol{x}^d$. By exactly the same calculation as the case (1), we can show that the derivatives of $\boldsymbol{x}_\pm^\ell$ and $\boldsymbol{x}^h$ at u_0 can be considered as linear transformations on the tangent space of $M = \boldsymbol{x}^d(U)$ at $p = \boldsymbol{x}^d(u_0)$. We respectively call $S_\pm^\ell(p) = -\mathrm{d}\boldsymbol{x}_\pm^\ell(u_0)$ and $S^h(p) = -\mathrm{d}\boldsymbol{x}^d(u_0)$ the *lightcone shape operator* and the *hyperbolic shape operator* of $M = \boldsymbol{x}^d(U)$ at $p = \boldsymbol{x}^d(u_0)$. We denote the eigenvalue of $S_\pm^\ell(p)$ by $\kappa_\pm^\ell(p)$ and the eigenvalue of $S^h(p)$ by $\kappa^h(p)$. By the relation $S_\pm^\ell(p) = S^h(p) \mp id_{T_pM}$, $S_\pm^\ell(p)$ and $S^h(p)$ have the same eigenvectors and we have a relation that $\kappa_\pm^\ell(p) = \kappa^h(p) \mp 1$.

We now define the notion of Gauss–Kronecker curvatures of $M = \boldsymbol{x}^d(U)$ at $p = \boldsymbol{x}(u_0)$ as follows:

$K_\ell^\pm(u_0) = \det S_\pm^\ell(p)$; The lightcone Gauss–Kronecker curvature,

$K_h(u_0) = \det S_p^h$; The hyperbolic Gauss–Kronecker curvature.

We say that a point $u_0 \in U$ or $p = \boldsymbol{x}^d(u_0)$ is an *umbilic point* if $S_\pm^\ell(p) = \kappa_\pm^\ell(p) id_{T_pM}$. Since the eigenvectors of $S_\pm^\ell(p)$ and $S^h(p)$ are the same, the above condition is equivalent to the condition $S^h(p) = \kappa^h(p) id_{T_pM}$. We say that $M = \boldsymbol{x}^d(U)$ is *totally umbilic* if all points on M are umbilic. Here, we consider the following model hypersurfaces in de Sitter space. We consider the intersection of S_1^n with a hyperplane in $\mathbb{R}_1^{n+1}$:

$$HS(\boldsymbol{n}, c) = HP(\boldsymbol{n}, c) \cap S_1^n.$$

We say that $HS(\boldsymbol{n}, c)$ is a *hyperbolic hyperquadric* if $\boldsymbol{n}$ is spacelike, a *parabolic hyperquadric* if $\boldsymbol{n}$ is lightlike and a *elliptic hyperquadric* if $\boldsymbol{n}$ is timelike. We can show the following classification of totally umbilic hypersurfaces in S_1^n by using exactly the same method as the proof of Proposition 2.2.

Proposition 2.3. *Suppose that* $M = \boldsymbol{x}^d(U) \subset S_1^n$ *is totally umbilic. Then* $\kappa_\pm^\ell(p)$ *is constant* $\kappa_\pm^\ell$*. Under this condition, we have the following classification.*

(1) *Suppose that* $(\kappa_\pm^\ell)^2 \pm 2\kappa_\pm^\ell \neq 0$.

(a) *If* $(\kappa_\pm^\ell)^2 \pm 2\kappa_\pm^\ell > 0$*,then* M *is a part of the hyperbolic hyperquadric*

$$HS\left(\boldsymbol{c}, \frac{\kappa_\pm^\ell \pm 1}{\sqrt{(\kappa_\pm^\ell)^2 \pm 2\kappa_\pm^\ell}}\right),$$

where

$$\boldsymbol{c} = \frac{1}{\sqrt{(\kappa_\pm^\ell)^2 \pm 2\kappa_\pm^\ell}}(\kappa_\pm^\ell \boldsymbol{x}^d(u) + \boldsymbol{x}_\pm^\ell(u)) \in S_1^n$$

is a constant spacelike vector.

(b) *If* $(\kappa_\pm^\ell)^2 \pm 2\kappa_\pm^\ell < 0$, *then* M *is a part of the elliptic hyperquadric*

$$HS\left(\boldsymbol{c}, \frac{\kappa_\pm^\ell \pm 1}{\sqrt{-(\kappa_\pm^\ell)^2 \mp 2\kappa_\pm^\ell}}\right),$$

where

$$\boldsymbol{c} = \frac{1}{\sqrt{-(\kappa_\pm^\ell)^2 \mp 2\kappa_\pm^\ell}}(\kappa_\pm^\ell \boldsymbol{x}^d(u) + \boldsymbol{x}_\pm^\ell(u)) \in H^n(-1)$$

is a constant timelike vector.

(2) *If* $(\kappa_\pm^\ell)^2 \pm 2\kappa_\pm^\ell = 0$, *then* M *is a part of the parabolic hyperquadric* $HS(\boldsymbol{c}, \kappa_\pm^\ell \pm 1)$, *where* $\boldsymbol{c} = \kappa_\pm^\ell \boldsymbol{x}^d(u) + \boldsymbol{x}_\pm^\ell(u) \in LC^*$ *is a constant lightlike vector.*

Case (3) We assume that $\boldsymbol{x}_+^\ell : U \longrightarrow LC^*$ is a spacelike embedding (i.e., an embedding and $(\boldsymbol{x}_+^\ell)_{u_i}$, $(i = 1, \ldots, n-1)$ are spacelike vectors). We call $\boldsymbol{x}^h(u_0)$ the *hyperbolic normal vector* to $M = \boldsymbol{x}_+^\ell(U)$ at $p = \boldsymbol{x}_+^\ell(u_0)$ and $\boldsymbol{x}^d(u_0)$ the *de Sitter normal vector* to $M = \boldsymbol{x}_+^\ell(U)$ at $p = \boldsymbol{x}_+^\ell(u_0)$. We call a mapping $\boldsymbol{x}_-^\ell : U \longrightarrow LC^*$ the *lightcone Gauss image* of $M = \boldsymbol{x}_+^\ell(U)$. We also respectively call $\boldsymbol{x}^h : U \longrightarrow H^n(-1)$ the *hyperbolic Gauss image* and $\boldsymbol{x}^d : U \longrightarrow S_1^n$ the *de Sitter Gauss image* of $M = \boldsymbol{x}_+^\ell(U)$. We investigated the extrinsic differential geometry of $M = \boldsymbol{x}_+^\ell(U)$ by using $\boldsymbol{x}_-^\ell, \boldsymbol{x}^h, \boldsymbol{x}^d$ like as the Gauss map of a hypersurface in Euclidean space, in [18]. For the purpose, we have shown that the derivatives $\mathrm{d}\boldsymbol{x}_-^\ell(u_0)$, $\mathrm{d}\boldsymbol{x}^h(u_0)$, $\mathrm{d}\boldsymbol{x}^d(u_0)$ can be considered as linear transformations on the tangent space T_pM where $p = \boldsymbol{x}_+^\ell(u_0)$. We respectively call the linear transformations $S^\ell(p) = -\mathrm{d}\boldsymbol{x}_-^\ell(u_0) : T_pM \longrightarrow T_pM$ the *lightcone shape operator*, $S^h(p) = -\mathrm{d}\boldsymbol{x}^h(u_0) : T_pM \longrightarrow T_pM$ the *hyperbolic shape operator* and $S^d(p) = -\mathrm{d}\boldsymbol{x}^d(u_0) : T_pM \longrightarrow T_pM$ the *de Sitter shape operator*. We respectively denote the eigenvalues of $S^\ell(p)$ by $\kappa^\ell(p)$, $S^h(p)$ by $\kappa^h(p)$ and $S^d(p)$ by $\kappa^d(p)$, which are respectively called the *lightcone principal curvature*, the *hyperbolic principal curvature* and the *de Sitter principal curvature* of M at p. We might consider that $\mathrm{d}\boldsymbol{x}_+^\ell(u_0)$ is the identity mapping on T_pM under the identification between U and M through $\boldsymbol{x}_+^\ell$. By the relations among $\boldsymbol{x}_+^\ell, \boldsymbol{x}_-^\ell, \boldsymbol{x}^h, \boldsymbol{x}^d$, the principal directions of $S^\ell(p)$, $S^h(p)$, $S^d(p)$ are the common and we have the following relations between the corresponding principal curvatures:

$$\kappa^h(p) = \frac{\kappa^\ell(p) - 1}{2} \quad \text{and} \quad \kappa^d(p) = \frac{-\kappa^\ell(p) - 1}{2}.$$

We now define the notion of curvatures of $M = \boldsymbol{x}_+^\ell(U)$ at $p = \boldsymbol{x}_+^\ell(u_0)$ as follows:

$K_\ell(u_0) = \det S^\ell(p)$; The lightcone Gauss–Kronecker curvature,
$K_h(u_0) = \det S^h(p)$; The hyperbolic Gauss–Kronecker curvature,
$K_d(u_0) = \det S^d(p)$; The de Sitter Gauss–Kronecker curvature.

We can define the notion of umbilicity like as the case of hypersurfaces in Euclidean space. We say that a point $p = \boldsymbol{x}_+^\ell(u_0)$ (or u_0) is an *umbilic point* if $S^\ell(p) = \kappa^\ell(p) id_{T_pM}$. Since the eigenvectors of $S^\ell(p)$, $S^h(p)$ and $S^d(p)$ are the same, the above condition is equivalent to both the conditions $S^h(p) = \kappa^h(p) id_{T_pM}$ and $S^d(p) = \kappa^d(p) id_{T_pM}$. We say that $M = \boldsymbol{x}_+^\ell(U)$ is *totally umbilic* if all points on M are umbilic. We now consider what is the totally umbilic hypersurface in the lightcone LC^*. We consider the intersection of LC^* with a hyperplane in $\mathbb{R}_1^{n+1}$:

$$HL(\boldsymbol{n}, c) = HP(\boldsymbol{n}, c) \cap LC^*.$$

We say that $HL(\boldsymbol{n}, c)$ is a *hyperbolic hyperquadric* if $\boldsymbol{n}$ is spacelike, a *parabolic hyperquadric* if $\boldsymbol{n}$ is lightlike and a *elliptic hyperquadric* if $\boldsymbol{n}$ is timelike. In [18] we showed the following classification of totally umbilic hypersurfaces in LC^*.

Proposition 2.4. *Suppose that $M = \boldsymbol{x}_+^\ell(U)$ is totally umbilic. Then $\kappa^\ell(p)$ is constant κ^ℓ. Under this condition, we have the following classification.*

(1) *If $\kappa^\ell < 0$, then M is a part of the hyperbolic hyperquadric $HL(\boldsymbol{c}, 1/\sqrt{-\kappa^\ell})$, where*

$$\boldsymbol{c} = \frac{-1}{2\sqrt{-\kappa^\ell}}(\kappa^\ell \boldsymbol{x}_+^\ell(u) + \boldsymbol{x}_-^\ell(u)) \in S_1^n$$

is a constant spacelike vector.

(2) *If $\kappa^\ell = 0$, then M is a part of the parabolic hyperquadric $HL(\boldsymbol{c}, -2)$, where $\boldsymbol{c} = \boldsymbol{x}_-^\ell(u) \in LC^*$ is a constant lightlike vector.*

(3) *If $\kappa^\ell > 0$, then M is a part of the elliptic hyperquadric $HL(\boldsymbol{c}, -1/\sqrt{\kappa^\ell})$, where*

$$\boldsymbol{c} = \frac{1}{2\sqrt{\kappa^\ell}}(\kappa^\ell \boldsymbol{x}_+^\ell(u) + \boldsymbol{x}_-^\ell(u)) \in H^n(-1)$$

is a constant timelike vector.

By the above proposition, we can classify the umbilic point as follows. Let $p = \boldsymbol{x}_+^\ell(u_0) \in M = \boldsymbol{x}_+^\ell(U)$ be an umbilic point; we say that p is a *timelike umbilic point* if $\kappa^\ell < 0$, a *lightlike umbilic point* (or *lightcone flat point*) if $\kappa^\ell = 0$, or a *spacelike umbilic point* if $\kappa^\ell > 0$.

In [18] we have shown the lightcone Weingarten formula. Since $(\boldsymbol{x}_+^\ell)_{u_i}$ $(i = 1, \ldots n-1)$ are spacelike vectors, we induce the Riemannian metric (the *lightcone first fundamental form*) $\mathrm{d}s^2 = \sum_{i=1}^{n-1} g_{ij}^\ell \mathrm{d}u_i \mathrm{d}u_j$ on $M = \boldsymbol{x}_+^\ell(U)$, where $g_{ij}^\ell(u) = \langle (\boldsymbol{x}_+^\ell)_{u_i}(u), (\boldsymbol{x}_+^\ell)_{u_j}(u) \rangle$ for any $u \in U$. We also define the *lightcone second fundamental invariant* by $h_{ij}^\ell(u) = \langle -(\boldsymbol{x}_-^\ell)_{u_i}(u), (\boldsymbol{x}_+^\ell)_{u_j}(u) \rangle$ for any $u \in U$.

Proposition 2.5. *Under the above notations, we have the following lightcone Weingarten formula:*

$$\left(\boldsymbol{x}_-^\ell\right)_{u_i} = -\sum_{j=1}^{n-1} \left(h^\ell\right)_i^j (\boldsymbol{x}_+^\ell)_{u_j},$$

where $\left(h^\ell\right)_i^j = \left(h_{ik}^\ell\right)\left(g_\ell^{kj}\right)$ *and* $\left(g_\ell^{kj}\right) = \left(g_{kj}^\ell\right)^{-1}$.

As a corollary of the above proposition, we have an explicit expression of the lightcone Gauss–Kronecker curvature by using Riemannian metric and the lightcone second fundamental invariant.

Corollary 2.6. *Under the same notations as in the above proposition, the lightcone Gauss–Kronecker curvature is given by*

$$K_\ell = \frac{\det\left(h_{ij}^\ell\right)}{\det\left(g_{\alpha\beta}\right)}.$$

We say that a point $p = \boldsymbol{x}(u)$ is a *lightcone parabolic point* if $K^\ell(u) = 0$, which is equivalent to the condition that $\det(h_{ij}^\ell)(u) = 0$.

3. Spacelike parallels and caustics in Minkowski pseudo-spheres

In this section we introduce the unified notion of parallels of a spacelike hypersurface in Minkowski pseudo-spheres. For any fixed real number $\phi \in \mathbb{R}$, we define a mapping $\mathcal{L}_1^\phi : U \longrightarrow \Delta_1$ by

$$\mathcal{L}_1^\phi(u) = \left(\frac{\exp(\phi)}{2}\boldsymbol{x}_+^\ell(u) + \frac{\exp(-\phi)}{2}\boldsymbol{x}_-^\ell(u),\ \frac{\exp(\phi)}{2}\boldsymbol{x}_+^\ell(u) - \frac{\exp(-\phi)}{2}\boldsymbol{x}_-^\ell(u)\right).$$

We respectively call the images of mappings

$$\pi_{11} \circ \mathcal{L}_1^\phi(u) = \frac{\exp(\phi)}{2}\boldsymbol{x}_+^\ell(u) + \frac{\exp(-\phi)}{2}\boldsymbol{x}_-^\ell(u)$$

the *hyperbolic parallel* and

$$\pi_{12} \circ \mathcal{L}_1^{\phi}(u) = \frac{\exp(\phi)}{2}\boldsymbol{x}_+^{\ell}(u) - \frac{\exp(-\phi)}{2}\boldsymbol{x}_-^{\ell}(u)$$

the *de Sitter parallel*.

We now explain why we call these images parallels. If $\phi = 0$, $\pi_{11} \circ \mathcal{L}_1^0(u) = \boldsymbol{x}^h(u)$ and $\pi_{12} \circ \mathcal{L}_1^0(u) = \boldsymbol{x}^d(u)$. Since we have the relations

$$\boldsymbol{x}_+^{\ell}(u) = \boldsymbol{x}^h(u) + \boldsymbol{x}^d(u), \qquad \boldsymbol{x}_-^{\ell}(u) = \boldsymbol{x}^h(u) - \boldsymbol{x}^d(u),$$

we can translate the hyperbolic parallels and the de Sitter parallels into

$$\pi_{11} \circ \mathcal{L}_1^{\phi}(u) = \cosh\phi\boldsymbol{x}^h(u) + \sinh\phi\boldsymbol{x}^d(u)$$

and

$$\pi_{12} \circ \mathcal{L}_1^{\phi}(u) = \sinh\phi\boldsymbol{x}^h(u) + \cosh\phi\boldsymbol{x}^d(u).$$

The above formula means that $\pi_{11} \circ \mathcal{L}_1^{\phi}(u)$ is the point on the geodesic started from $\boldsymbol{x}^h(u)$ directed by $\boldsymbol{x}^d(u)$. Therefore the image of $\pi_{11} \circ \mathcal{L}_1^{\phi}$ is the locus of the points on the geodesics from $\boldsymbol{x}^h(U)$ directed by the unit normals $\boldsymbol{x}^d$ with a constant length. The second formula also means that the geodesics starts from $\boldsymbol{x}^d(U)$ directed by the unit normals $\boldsymbol{x}^h$ with a constant length. Therefore we might call these parallels.

We also consider extra properties of $\mathcal{L}_1^{\phi} : U \longrightarrow \Delta_1$ from the viewpoint of the contact geometry. For positive real numbers λ, μ with $\lambda \cdot \mu = 1$, we define a diffeomorphism

$$\Psi_{(\lambda,\mu)} : \mathbb{R}_1^{n+1} \times \mathbb{R}_1^{n+1} \longrightarrow \mathbb{R}_1^{n+1} \times \mathbb{R}_1^{n+1}$$

by

$$\Psi_{(\lambda,\mu)}(\boldsymbol{v}, \boldsymbol{w}) = \left(\frac{\lambda\boldsymbol{v} + \mu\boldsymbol{w}}{2}, \frac{\lambda\boldsymbol{v} - \mu\boldsymbol{w}}{2}\right).$$

Since $\lambda \cdot \mu = 1$, we have $\Psi_{(\lambda,\mu)}(\Delta_4) = \Delta_1$. We also have

$$\Psi_{(\lambda,\mu)}^*\theta_{12} = \left\langle \frac{\lambda\boldsymbol{v} + \mu\boldsymbol{w}}{2}, d\left(\frac{\lambda\boldsymbol{v} - \mu\boldsymbol{w}}{2}\right)\right\rangle = -\frac{1}{2}\theta_{42},$$

so that $\Psi_{(\lambda,\mu)}|\Delta_4$ is a contact diffeomorphism. By definition, we have $\Psi_{(\exp(\phi),\exp(-\phi))} \circ \mathcal{L}_4 = \mathcal{L}_1^{\phi}$. Since $\mathcal{L}_4$ is a Legendrian embedding, $\mathcal{L}_1^{\phi}$ is a Legendrian embedding.

Proposition 3.1. *The hyperbolic parallel (respectively, de Sitter parallel) is the wave front set of the Legendrian mapping $\pi_{11} \circ \mathcal{L}_1^{\phi}$ (respectively, $\pi_{12} \circ \mathcal{L}_1$).*

We call $\mathcal{L}_1^{\phi}$ a *Legendrian parallel.*

In the classical Euclidean case, if the distance of the parallels varies, the locus of the singularities of parallels forms a caustics of a certain Lagrangian manifold.

We now review some properties of symplectic manifolds and Lagrangian submanifolds. Let N be a $2n$-dimensional smooth manifold and ω be a 2-form on N. The 2-form ω is *non-degenerate* if $(\omega)^n \neq 0$ at any point of N. We say that (N, ω) is a *symplectic manifold* if ω is a closed non-degenerate 2-form. In this case ω is called a *symplectic structure* or a *symplectic form*. Let $\phi : N \longrightarrow N'$ be a diffeomorphism between symplectic manifolds (N, ω) and (N', ω'). We say that ϕ is a *symplectic diffeomorphism* if $\phi^*\omega' = \omega$. Two symplectic manifolds (N, ω) and (N', ω') are *symplectic diffeomorphic* if there exists a symplectic diffeomorphism $\phi : N \longrightarrow N'$. A submanifold $i : L \subset N$ of a symplectic manifold (N, ω) is said to be *Lagrangian* if $\dim L = n$ and $i^*\omega = 0$. We say that a smooth fiber bundle $\pi : E \longrightarrow M$ is called a *Lagrangian fibration* if its total space E is furnished with a symplectic structure and its fibers are Lagrangian submanifolds. Let $\pi : E \longrightarrow M$ be a Lagrangian fibration. For a Lagrangian submanifold $i : L \subset E$, $\pi \circ i : L \longrightarrow M$ is called a *Lagrangian map*. The critical value set of the Lagrangian map $\pi \circ i$ is called a *caustics* of

i which is denoted by C_L. For any $p \in E$, it is known that there is a local coordinate system $(x_1, \ldots, x_m, p_1, \ldots, p_m)$ around p such that

$$\pi(x_1, \ldots, x_m, p_1, \ldots, p_m) = (x_1, \ldots, x_m)$$

and the symplectic form is given by

$$\omega = \sum_{i=1}^{m} \mathrm{d}p_i \wedge \mathrm{d}x_i$$

(cf. [1], 20.3).

We now consider what is the corresponding caustics for spacelike parallels in Minkowski pseudo-spheres. We consider the symplectification $(\Delta_1 \times \mathbb{R}_+, -\mathrm{d}(\eta\theta_{12})) = (\Delta_1 \times \mathbb{R}_+, \mathrm{d}(\eta\theta_{11}))$ of the contact manifold Δ_1, where $((\boldsymbol{v}, \boldsymbol{w}), \eta) \in \Delta_1 \times \mathbb{R}_+$. Here $\mathbb{R}_+$ is a set of the positive real numbers. Define a mapping

$$\widetilde{\mathcal{L}}_1 : U \times \mathbb{R} \longrightarrow \Delta_1 \times \mathbb{R}_+$$

by

$$\widetilde{\mathcal{L}}_1(u, \phi) = (\mathcal{L}_1^{\phi}(u), \exp(-\phi)).$$

Since $\mathcal{L}_1^{\phi}$ is an embedding for any fixed ϕ, $\widetilde{\mathcal{L}}_1$ is also an embedding. By a direct calculation, we have $(\widetilde{\mathcal{L}}_1)^*(\eta\theta_{12}) = -\exp(-\phi)\mathrm{d}\phi$, so that $\mathrm{d}(\widetilde{\mathcal{L}}_1)^*(\eta\theta_{12}) = -\mathrm{d}\exp(-\phi) \wedge \mathrm{d}\phi = 0$. This means that $\widetilde{\mathcal{L}}_1$ is a Lagrangian embedding. Let $\widetilde{\pi}_{11} : \Delta_1 \times \mathbb{R}_+ \longrightarrow H^n(-1)$ and $\widetilde{\pi}_{12} : \Delta_1 \times \mathbb{R}_+ \longrightarrow S_1^n$ be the canonical projections, then both the projections are Lagrangian fibrations. Therefore, we have two Lagrangian mappings:

$$\widetilde{\pi}_{11} \circ \widetilde{\mathcal{L}}_1 : U \times \mathbb{R} \longrightarrow H^n(-1); \qquad \widetilde{\pi}_{11} \circ \widetilde{\mathcal{L}}_1(u, \phi) = \pi_{11} \circ \mathcal{L}_1^{\phi}(u)$$
$$\widetilde{\pi}_{12} \circ \widetilde{\mathcal{L}}_1 : U \times \mathbb{R} \longrightarrow S_1^n; \qquad \widetilde{\pi}_{12} \circ \widetilde{\mathcal{L}}_1(u, \phi) = \pi_{12} \circ \mathcal{L}_1^{\phi}(u).$$

By definition, we have

$$\frac{\partial(\widetilde{\pi}_{11} \circ \widetilde{\mathcal{L}}_1)}{\partial u_i}(u, \phi) = \frac{\partial(\pi_{11} \circ \mathcal{L}_1^{\phi})}{\partial u_i}(u), \qquad \frac{\partial(\widetilde{\pi}_{11} \circ \widetilde{\mathcal{L}}_1)}{\partial \phi}(u, \phi) = \pi_{12} \circ \mathcal{L}_1^{\phi}(u),$$
$$\frac{\partial(\widetilde{\pi}_{12} \circ \widetilde{\mathcal{L}}_1)}{\partial u_i}(u, \phi) = \frac{\partial(\pi_{12} \circ \mathcal{L}_1^{\phi})}{\partial u_i}(u), \qquad \frac{\partial(\widetilde{\pi}_{12} \circ \widetilde{\mathcal{L}}_1)}{\partial \phi}(u, \phi) = \pi_{11} \circ \mathcal{L}_1^{\phi}(u)$$

for $i = 1, \ldots, n-1$. Since $\mathcal{L}_1$ is a Legendrian embedding, we have $\langle(\pi_{11} \circ \mathcal{L}_1^{\phi})_{u_i}(u), \pi_{12} \circ \mathcal{L}_1^{\phi}(u)\rangle = \langle \pi_{11} \circ \mathcal{L}_1^{\phi}(u), (\pi_{12} \circ \mathcal{L}_1^{\phi})_{u_i}(u)\rangle = 0$. It follows that $(\pi_{11} \circ \mathcal{L}_1^{\phi})_{u_1}(u), \ldots, (\pi_{11} \circ \mathcal{L}_1^{\phi})_{u_{n-1}}(u)$ is linearly independent if and only if $(\widetilde{\pi}_{11} \circ \widetilde{\mathcal{L}}_1)_{u_1}(u, \phi), \ldots, (\widetilde{\pi}_{11} \circ \widetilde{\mathcal{L}}_1)_{u_{n-1}}(u, \phi), (\widetilde{\pi}_{11} \circ \widetilde{\mathcal{L}}_1)_{\phi}(u, \phi)$ is linearly independent. Therefore, $(u, \phi) \in U \times \mathbb{R}$ is a singular point of $\widetilde{\pi}_{11} \circ \widetilde{\mathcal{L}}_1$ if and only if u is a singular point of $\pi_{11} \circ \mathcal{L}_1^{\phi}$. The same assertion holds for $\widetilde{\pi}_{12} \circ \widetilde{\mathcal{L}}_1$. We denote the critical value sets of $\widetilde{\pi}_{11} \circ \widetilde{\mathcal{L}}_1$ by $C_h(\widetilde{\mathcal{L}}_1)$ and call the *hyperbolic caustics* of $\widetilde{\mathcal{L}}_1$. We also denote the critical value sets of $\widetilde{\pi}_{12} \circ \widetilde{\mathcal{L}}_1$ by $C_d(\widetilde{\mathcal{L}}_1)$ and call the *de Sitter caustics* of $\widetilde{\mathcal{L}}_1$. The above arguments show the following proposition.

Proposition 3.2. *The hyperbolic caustics $C_h(\widetilde{\mathcal{L}}_1)$ (respectively, de Sitter caustics $C_d(\widetilde{\mathcal{L}}_1)$) is the locus of singularities of the hyperbolic parallels (respectively, de Sitter parallels).*

4. Caustics and evolutes of spacelike hypersurfaces in Minkowski pseudo-spheres

We now introduce the notion of evolutes of spacelike hypersurfaces in the lightcone. For a spacelike embedding $\boldsymbol{x}_+^{\ell} : U \longrightarrow LC^*$, we define the *total evolute* of $M = \boldsymbol{x}_+^{\ell}(U)$ by

$$TE_M = \left\{ \frac{|\kappa^\ell(u)|}{2\sqrt{|\kappa^\ell(u)|}} \left(\boldsymbol{x}^\ell_+(u) + \frac{1}{\kappa^\ell(u)} \boldsymbol{x}^\ell_-(u) \right) \middle| \kappa^\ell(u) \text{ is a lightcone principal curvature at } p = \boldsymbol{x}^\ell_+(u),\ u \in U \right\}.$$

For a spacelike hypersurface as the above, we have the following decomposition of the total evolute:

$$TE_M(u) = HE_M \cup DE_M,$$

where

$$HE_M = \left\{ \frac{1}{2\sqrt{\kappa^\ell(u)}} (\kappa^\ell(u)\boldsymbol{x}^\ell_+(u) + \boldsymbol{x}^\ell_-(u)) | \kappa^\ell(u) \text{ is a lightcone principal curvature with } \kappa^\ell(u) > 0 \text{ at } p = \boldsymbol{x}^\ell_+(u),\ u \in U \right\}$$

and

$$DE_M = \left\{ \frac{-1}{2\sqrt{-\kappa^\ell(u)}} (\kappa^\ell(u)\boldsymbol{x}^\ell_+(u) + \boldsymbol{x}^\ell_-(u)) | \kappa^\ell(u) \text{ is a lightcone principal curvature with } \kappa^\ell(u) < 0 \text{ at } p = \boldsymbol{x}^\ell_+(u),\ u \in U \right\}.$$

We can show that $HE_M \subset H^n(-1)$ and $DE_M \subset S^n_1$. Therefore we call HE_M (respectively, DE_M) the *hyperbolic evolute* (respectively, *de Sitter evolute*) of $M = \boldsymbol{x}^\ell_+(U)$.

For any fixed lightcone principal curvature κ^ℓ, we define a smooth mapping $HE^{\kappa^\ell}_M : U_+ \longrightarrow H^n(-1)$ by

$$HE^{\kappa^\ell}_M(u) = \frac{1}{2\sqrt{\kappa^\ell(u)}} (\kappa^\ell(u)\boldsymbol{x}^\ell_+(u) + \boldsymbol{x}^\ell_-(u)),$$

where $U_+ = \{u \in U \mid \kappa^\ell(u) > 0\}$. We can also define a smooth mapping $SE^{\kappa^\ell}_M : U_- \longrightarrow S^n_1$ by the similar way for $U_- = \{u \in U \mid \kappa^\ell(u) < 0\}$. The above mappings give local parametrizations of the evolutes. We have the following proposition:

Proposition 4.1. *Let $M = \boldsymbol{x}^\ell_+(U)$ be a spacelike hypersurface in LC^* without lightcone parabolic points and lightcone flat points.*

(A) *The following are equivalent:*
  (1) *M is totally umbilic with $\kappa^\ell > 0$.*
  (2) *HE_M is a point in $H^n_+(-1)$.*
  (3) *M is a part of an elliptic hyperquadric.*

(B) *The following are equivalent:*
  (1) *M is totally umbilic with $\kappa^\ell < 0$.*
  (2) *DE_M is a point in S^n_1.*
  (3) *M is a part of an hyperbolic hyperquadric.*

Proof. (A) By Proposition 2.4, (1) and (3) are equivalent.

We assume that the condition (1) holds, then the lightcone principal curvature $\kappa_\ell(u) = \kappa_\ell$ is constant and $\kappa_\ell > 0$. Therefore we have

$$\frac{\partial HE^{\kappa^\ell}_M}{\partial u_i}(u) = \frac{1}{2\sqrt{\kappa^\ell}} (\kappa^\ell(\boldsymbol{x}^\ell_+)_{u_i}(u) + (\boldsymbol{x}^\ell_-)_{u_i}(u))$$

for any $u \in U$. By the definition of the lightcone principal curvature, $-(\boldsymbol{x}^\ell_-)_{u_i} = \kappa_\ell(\boldsymbol{x}^\ell_+)_{u_i}$ for $i = 1, \ldots, n-1$. It follows that $\partial(HE^{\kappa_\ell}_M/\partial u_i)(u) = 0$ for $i = 1, \ldots, n-1$. It concludes that $HE^{\kappa_\ell}_M(u)$ is a point.

On the other hand, we calculate that

$$\frac{\partial HE^{\kappa^\ell}_M}{\partial u_i}(u) = \frac{1}{2}\left\{\frac{\kappa^\ell_{u_i}(u)}{2\sqrt{\kappa^\ell(u)}}\left(\boldsymbol{x}^\ell_+(u) - \frac{1}{\kappa^\ell(u)}\boldsymbol{x}^\ell_-(u)\right) + \sqrt{\kappa^\ell(u)}\left((\boldsymbol{x}^\ell_+)_{u_i}(u) + \frac{1}{\kappa^\ell(u)}(\boldsymbol{x}^\ell_-)_{u_i}(u)\right)\right\}.$$

By the lightcone Weingarten formula (Proposition 2.5), we have

$$\frac{\partial HE^{\kappa^\ell}_M}{\partial u_i}(u) = \frac{1}{2}\left\{\frac{\kappa^\ell_{u_i}(u)}{2\sqrt{\kappa^\ell(u)}}\left(\boldsymbol{x}^\ell_+(u) - \frac{1}{\kappa^\ell(u)}\boldsymbol{x}^\ell_-(u)\right) + \sqrt{\kappa^\ell(u)}\left(\sum_{j=1}^{n-1}\left(\delta_{ij} - \frac{1}{\kappa^\ell(u)}(h^\ell)^j_i\right)(\boldsymbol{x}^\ell_+)_{u_j}(u)\right)\right\}.$$

Since $\{\boldsymbol{x}^\ell_+, \boldsymbol{x}^\ell_-, (\boldsymbol{x}^\ell_+)_{u_1}, \ldots, (\boldsymbol{x}^\ell_+)_{u_{n-1}}\}$ is linearly independent, $(\partial HE^{\kappa^\ell}_M/\partial u_i)(u) = 0$ if and only if M is umbilic at $p = \boldsymbol{x}^\ell_+(u)$ and $\kappa^\ell_{u_i}(u) = 0$ for $i = 1, \ldots, n-1$. It follows that (1) and (2) are equivalent. This completes the proof of (A).

The assertion (B) also follows from straightforward calculations like as those for the proof of (A). □

In [14] we have defined the notion of evolutes of hypersurfaces in $H^n(-1)$ as follows. For an embedding $\boldsymbol{x}^h : U \longrightarrow H^n(-1)$, we define the *total evolute* of $M = \boldsymbol{x}^h(U)$ by

$$TE^{\pm}_M = \left\{\pm\frac{\kappa^d(u)}{\sqrt{|(\kappa^d)^2(u) - 1|}}\left(\boldsymbol{x}^h(u) + \frac{1}{\kappa^d(u)}\boldsymbol{x}^d(u)\right)\middle|\, \kappa^d(u) \text{ is a de Sitter principal curvature at } p = \boldsymbol{x}^h(u),\ u \in U\right\}.$$

By the relations $\boldsymbol{x}^\ell_\pm = \boldsymbol{x}^h \pm \boldsymbol{x}^d$ and $\kappa^\ell_\pm(u) = -1 \pm \kappa^d(u)$, the above definition of the total evolute for an embedding $\boldsymbol{x}^h : U \longrightarrow H^n(-1)$ is the same as the definition of the total evolute for $\boldsymbol{x}^\ell_+$. We can also define the total evolute for a spacelike embedding $\boldsymbol{x}^d : U \longrightarrow S^n_1$. It also coincides with the definition of the total evolute for $\boldsymbol{x}^\ell_+$. Therefore we omit the detail here.

5. Timelike and spacelike height functions

In this section we consider two kinds of families of height functions on a spacelike hypersurface in the lightcone in order to describe the hyperbolic evolute and the de Sitter evolute of the spacelike hypersurface.

For the purpose, we need some concepts and results in the theory of unfoldings of function germs. We shall give a brief review of the theory in Appendices A and B.

We now define two families of functions

$$H^T : U \times H^n(-1) \longrightarrow \mathbb{R}$$

by $H^T(u, \boldsymbol{v}) = \langle \boldsymbol{x}^\ell_+(u), \boldsymbol{v}\rangle$ and

$$H^S : U \times S^n_1 \longrightarrow \mathbb{R}$$

by $H^S(u, \boldsymbol{v}) = \langle \boldsymbol{x}^\ell_+(u), \boldsymbol{v}\rangle$. We call H^T (respectively, H^S) a *timelike height function* (respectively, a *spacelike height function*) on $\boldsymbol{x}^\ell_+ : U \longrightarrow LC^*$. We denote that $h^T_v(u) = H^T(u, \boldsymbol{v})$ (respectively, $h^S_v(u) = H^S(u, \boldsymbol{v})$).

Proposition 5.1. *Let $\boldsymbol{x}^\ell_+ : U \longrightarrow LC^*$ be a spacelike embedding. Then*

(1) $(\partial h^T_v/\partial u_i)(u) = 0$ $(i = 1, \ldots, n-1)$ *if and only if there exists a non-zero real number λ such that $\boldsymbol{v} = \lambda\boldsymbol{x}^\ell_+(u) + (1/4\lambda)\boldsymbol{x}^\ell_-(u)$.*
(2) $(\partial h^S_v/\partial u_i)(u) = 0$ $(i = 1, \ldots, n-1)$ *if and only if there exists a non-zero real number λ such that $\boldsymbol{v} = \lambda\boldsymbol{x}^\ell_+(u) - (1/4\lambda)\boldsymbol{x}^\ell_-(u)$.*

Proof. (1) There exist real numbers λ, μ, ξ_i $(i = 1, \ldots, n-1)$ such that $\boldsymbol{v} = \lambda \boldsymbol{x}_+^\ell + \mu \boldsymbol{x}_-^\ell + \sum_{i=1}^{n-1} \xi_i (\boldsymbol{x}_+^\ell)_{u_i}$. Since $(\partial H/\partial u_i)(u, \boldsymbol{v}) = \langle (\boldsymbol{x}_+^\ell)_{u_i}, \boldsymbol{v} \rangle$, we have $0 = \langle (\boldsymbol{x}_+^\ell)_{u_i}, \boldsymbol{v} \rangle = \sum_{j=1}^{n-1} \xi_j g_{ij}^\ell(u)$. Since g_{ij}^ℓ is positive definite, we have $\xi_j = 0$ $(j = 1, \ldots, n-1)$. We also have $-1 = \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 2\lambda\mu \langle \boldsymbol{x}_+^\ell, \boldsymbol{x}_-^\ell \rangle = -4\lambda\mu$. This completes the proof for the assertion (1). The proof for the assertion (2) is given by almost the same calculations as those for the assertion (1), so that we omit the detail. □

By Proposition 5.1, we can detect both of the catastrophe sets (cf. Appendix A) of H^T and H^S as follows:

$$C(H^T) = \left\{ (u, \boldsymbol{v}) \in U \times H^n(-1) | \boldsymbol{v} = \lambda \boldsymbol{x}_+^\ell(u) + \frac{1}{4\lambda} \boldsymbol{x}_-^\ell(u) \right\},$$

$$C(H^S) = \left\{ (u, \boldsymbol{v}) \in U \times S_1^n | \boldsymbol{v} = \lambda \boldsymbol{x}_+^\ell(u) - \frac{1}{4\lambda} \boldsymbol{x}_-^\ell(u) \right\}.$$

Here, we have the following decompositions:

$$C(H^T) = C_+(H^T) \cup C_-(H^T) \quad \text{and} \quad C(H^S) = C_+(H^S) \cup C_-(H^S),$$

where $C_+(H^T) = \{(u, \boldsymbol{v}) \mid \boldsymbol{v} = \lambda \boldsymbol{x}_+^\ell(u) + (1/4\lambda)\boldsymbol{x}_-^\ell(u),\ \lambda > 0\}$, $C_-(H^T) = \{(u, \boldsymbol{v}) \mid \boldsymbol{v} = \lambda \boldsymbol{x}_+^\ell(u) + (1/4\lambda)\boldsymbol{x}_-^\ell(u),\ \lambda < 0\}$ and the definitions of $C_+(H^S)$ and $C_-(H^S)$ are given by the similar way. We also calculate that

$$\frac{\partial^2 H^T}{\partial u_i \partial u_j}(u, \boldsymbol{v}) = \langle (\boldsymbol{x}_+^\ell)_{u_i u_j}(u), \boldsymbol{v} \rangle = -\lambda g_{ij}^\ell + \frac{1}{4\lambda} h_{ij}^\ell$$

on $C(H^T)$ and

$$\frac{\partial^2 H^S}{\partial u_i \partial u_j}(u, \boldsymbol{v}) = \langle (\boldsymbol{x}_+^\ell)_{u_i u_j}(u), \boldsymbol{v} \rangle = -\lambda g_{ij}^\ell - \frac{1}{4\lambda} h_{ij}^\ell$$

on $C(H^S)$.

Therefore, $\det(\mathcal{H}(h_v^T)(u)) = \det((\partial^2 H^T/\partial u_i \partial u_j)(u, \boldsymbol{v})) = 0$ (respectively, $\det(\mathcal{H}(h_v^S)(u)) = 0$) if and only if $\kappa^\ell(u) = 4\lambda^2$ (respectively, $\kappa^\ell(u) = -4\lambda^2$) is a lightcone principal curvature. Since $\boldsymbol{v} \in H^n(-1)$ (respectively, $\boldsymbol{v} \in S_1^n$) and $\kappa^\ell(u) = 4\lambda^2$ (respectively, $\kappa^\ell(u) = -4\lambda^2$) is a lightcone principal curvature with $\kappa^\ell(u) > 0$ (respectively, $\kappa^\ell(u) < 0$), we have

$$\mathcal{B}_{H^T} = HE_M \cup (-HE_M) \quad \text{(respectively, } \mathcal{B}_{H^S} = DE_M \cup (-DE_M)\text{)},$$

where $(-HE_M) = \{-\boldsymbol{v} \mid \boldsymbol{v} \in HE_M\}$ (respectively, $(-DE_M) = \{-\boldsymbol{v} \mid \boldsymbol{v} \in DE_M\}$).

Proposition 5.2. *We assume that $p = \boldsymbol{x}_+^\ell(u_0)$ is not a lightcone flat point of $M = \boldsymbol{x}_+^\ell(U)$, then we have the following assertions:*

(1) *p is an umbilic point with $\kappa^\ell(p) > 0$ if and only if there exists $\boldsymbol{v}_0 \in H^n(-1)$ such that u_0 is a singular point of $h_{v_0}^T$ and* $\operatorname{rank} \mathcal{H}(h_{v_0}^T)(u_0) = 0$.

(2) *p is an umbilic point with $\kappa^\ell(p) < 0$ if and only if there exists $\boldsymbol{v}_0 \in S_1^n$ such that u_0 is a singular point of $h_{v_0}^S$ and* $\operatorname{rank} \mathcal{H}(h_{v_0}^S)(u_0) = 0$.

Proof. (1) Since p is an umbilic point, $S_p^\ell = \kappa^\ell(p) id_{T_pM}$. There exists an orthogonal matrix Q such that ${}^tQ((h^\ell)_i^j)Q = \kappa^\ell(p)I$. Hence, we may consider the case $(h^\ell)_i^j = \kappa^\ell(p)I$, so that $(h_{ij}^\ell) = \kappa^\ell(p)(g_{ij}^\ell)$. Then we put $\boldsymbol{v}_0 = \lambda \boldsymbol{x}_+^\ell(u_0) + \mu \boldsymbol{x}_-^\ell(u_0) \in H^n(-1)$, where $\lambda = \pm(\kappa^\ell(p)/2\sqrt{\kappa^\ell(p)})$, $\mu = \pm(1/2\sqrt{\kappa^\ell(p)})$. In this case the Hessian matrix $\mathcal{H}(h_{v_0}^T)(u_0) = (-\lambda g_{ij}^\ell + \mu h_{ij}^\ell) = (-\lambda + \mu\kappa^\ell(p))(g_{ij}^\ell) = 0$.

On the other hand, if $-\lambda g_{ij}^\ell + \mu h_{ij}^\ell = 0$ for all i, j, then $(h_{ij}^\ell) = \kappa^\ell(p)(g_{ij}^\ell)$ $(\kappa^\ell(p) = \lambda/\mu)$. This is equivalent to the condition $((h^\ell)_i^j) = \kappa_p I$.

The proof of (2) is also given by direct calculations like as those of (1). □

We say that u_0 is a *timelike ridge point* (respectively, *spacelike ridge point*) if h_v^T (respectively, h_v^S) has the $A_{k\geq 3}$-type singular point at u_0, where $\boldsymbol{v} \in \mathcal{B}_{H^T}$ (respectively, $\boldsymbol{v} \in \mathcal{B}_{H^S}$).

For a function germ $f : (\mathbb{R}^{n-1}, \widetilde{u}_0) \longrightarrow \mathbb{R}$, f has A_k-type singular point at $\widetilde{u}_0$ if f is $\mathcal{R}^+$-equivalent to the germ $\pm u_1^{k+1} \pm u_2^2 \pm \cdots \pm u_{n-1}^2$. We say that two function germs $f_i : (\mathbb{R}^{n-1}, \widetilde{u}_i) \longrightarrow \mathbb{R}$ $(i = 1, 2)$ are $\mathcal{R}^+$-*equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^{n-1}, \widetilde{u}_1) \longrightarrow (\mathbb{R}^{n-1}, \widetilde{u}_2)$ and a real number c such that $f_2 \circ \Phi(u) = f_1(u) + c$.

We now consider the geometric meaning of ridge points. Let $F : LC^* \longrightarrow \mathbb{R}$ be a function and $\boldsymbol{x}_+^\ell : U \longrightarrow LC^*$ be a spacelike hypersurface. We say that $\boldsymbol{x}_+^\ell$ and $F^{-1}(0)$ have a *corank r contact* at $p_0 = \boldsymbol{x}(u_0)$ if the Hessian of the function $g(u) = F \circ \boldsymbol{x}_+^\ell(u)$ has corank r at u_0. We also say that $\boldsymbol{x}_+^\ell$ and $F^{-1}(0)$ have an A_k-*type contact* at $p_0 = \boldsymbol{x}(u_0)$ if the function $g(u) = F \circ \boldsymbol{x}(u)$ has the A_k-type singularity at u_0. By definition, if $\boldsymbol{x}_+^\ell$ and $F^{-1}(0)$ have an A_k-*type contact* at $p_0 = \boldsymbol{x}(u_0)$, then these have a corank 1 contact. For any $r \in \mathbb{R}$ and $\boldsymbol{a}_0 \in H_+^n(-1)$ (respectively, $\boldsymbol{a}_0 \in S_1^n$), we consider a function $F : H_+^n(-1) \longrightarrow \mathbb{R}$ defined by $F(\boldsymbol{u}) = \langle \boldsymbol{u}, \boldsymbol{a}_0 \rangle - r$. We denote that

$$HL(\boldsymbol{a}_0, r) = F^{-1}(0) = \{\boldsymbol{u} \in LC^* | \langle \boldsymbol{u}, \boldsymbol{a}_0 \rangle = r\}.$$

Then $HL(\boldsymbol{a}_0, r)$ is an elliptic hyperquadric (respectively, a hyperbolic hyperquadric) with center $\boldsymbol{a}_0$ if $\boldsymbol{a}_0$ is in $H_+^n(-1)$ (respectively, S_1^n). We put $\boldsymbol{a}_0 = HE_M^{\kappa^\ell}(u_0)$ (respectively $\boldsymbol{a}_0 = DE_M^{\kappa^\ell}(u_0)$) and $r_0 = -(\sqrt{|\kappa^\ell(u_0)|}/\kappa^\ell(u_0))$, where we fix a lightcone principal curvature $\kappa^\ell(u)$ on U around u_0, then we have the following simple proposition:

Proposition 5.3. *Under the above notations, there exists an integer k with* $1 \leq k \leq n-1$ *such that* $M = \boldsymbol{x}_+^\ell(U)$ *and* $HL(\boldsymbol{a}_0, r_0)$ *have corank k contact at* u_0.

In the above proposition, $HL(\boldsymbol{a}_0, r_0)$ is called an *osculating elliptic hyperquadric* (respectively, *osculating hyperbolic hyperquadric*) of $M = \boldsymbol{x}_+^\ell(U)$ if $\boldsymbol{a}_0 \in H_+^n(-1)$ (respectively, $\boldsymbol{a}_0 \in S_1^n$). We also call $\boldsymbol{a}_0$ the *center of the lightcone principal curvature* $\kappa^\ell(u_0)$. By Proposition 5.2, $M = \boldsymbol{x}_+^\ell(U)$ and the osculating elliptic hyperquadric (respectively, hyperbolic hyperquadric) have corank $n-1$ contact at an umbilic point. Therefore the hyperbolic (respectively, de Sitter) ridge point is not an umbilic point.

By the general theory of unfoldings of function germs, the bifurcation set $\mathcal{B}_F$ is non-singular at the origin if and only if the function $f = F|\mathbb{R}^n \times \{0\}$ has the A_2-type singularity (i.e., the fold type singularity). Therefore we have the following proposition:

Proposition 5.4. *Under the same notations as in the previous proposition, the total evolute* TE_M *is non-singular at* $\boldsymbol{a}_0 = TE_M^{\kappa^\ell}(u_0)$ *if and only if* $M = \boldsymbol{x}_+^\ell(U)$ *and* $HL(\boldsymbol{a}_0, r_0)$ *have* A_2*-type contact at* u_0. *Here,* $TE_M^{\kappa^\ell}(u_0) = HE_M^{\kappa^\ell}(u_0)$ *if* $\boldsymbol{a}_0 \in H_+^n(-1)$ *and* $TE_M^{\kappa^\ell}(u_0) = DE_M^{\kappa^\ell}(u_0)$ *if* $\boldsymbol{a}_0 \in S_1^n$.

6. Evolutes as caustics

In this section we naturally interpret the hyperbolic evolute and the de Sitter evolute of spacelike hypersurface in the lightcone as the caustics given in Section 3.

For a spacelike embedding $\boldsymbol{x}_+^\ell : U \longrightarrow LC^*$, we consider the timelike height function H^T and the spacelike height function H^S (cf. Section 5). We have the following proposition:

Proposition 6.1. *Both the timelike height function* $H^T : U \times H^n(-1) \longrightarrow \mathbb{R}$ *and the spacelike height function* $H^S : U \times S_1^n \longrightarrow \mathbb{R}$ *on* $\boldsymbol{x}_+^\ell$ *are Morse families of functions.*

Proof. First we consider the timelike height function.

For any $\boldsymbol{v} = (v_0, v_1, \ldots, v_n) \in H^n(-1)$, we have $v_0 = \pm\sqrt{v_1^2 + \cdots + v_n^2 + 1}$, so that

$$H^T(u, \boldsymbol{v}) = \mp x_0(u)\sqrt{v_1^2 + \cdots + v_n^2 + 1} + x_1(u)v_1 + \cdots + x_n(u)v_n,$$

where $\boldsymbol{x}_+^\ell(u) = (x_0(u), \ldots, x_n(u))$. We will prove that the mapping

$$\Delta H^T = \left(\frac{\partial H^T}{\partial u_1}, \ldots, \frac{\partial H^T}{\partial u_{n-1}}\right)$$

is non-singular at any point. The Jacobian matrix of ΔH^T is given as follows:

$$\begin{pmatrix} \langle (\boldsymbol{x}_+^\ell)_{u_1u_1}, \boldsymbol{\nu}\rangle & \cdots & \langle (\boldsymbol{x}_+^\ell)_{u_1u_{n-1}}, \boldsymbol{\nu}\rangle & -x_{0u_1}\dfrac{v_1}{v_0} + x_{1u_1} & \cdots & -x_{0u_1}\dfrac{v_n}{v_0} + x_{nu_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle (\boldsymbol{x}_+^\ell)_{u_{n-1}u_1}, \boldsymbol{\nu}\rangle & \cdots & \langle (\boldsymbol{x}_+^\ell)_{u_{n-1}u_{n-1}}, \boldsymbol{\nu}\rangle & -x_{0u_{n-1}}\dfrac{v_1}{v_0} + x_{1u_{n-1}} & \cdots & -x_{0u_{n-1}}\dfrac{v_n}{v_0} + x_{nu_{n-1}} \end{pmatrix},$$

where $(\boldsymbol{x}_+^\ell)_{u_iu_j} = \partial^2\boldsymbol{x}_+^\ell/\partial u_i\partial u_j$. We will show that the rank of the matrix

$$X = \begin{pmatrix} -x_{0u_1}\dfrac{v_1}{v_0} + x_{1u_1} & \cdots & -x_{0u_1}\dfrac{v_n}{v_0} + x_{nu_1} \\ \vdots & \vdots & \vdots \\ -x_{0u_{n-1}}\dfrac{v_1}{v_0} + x_{1u_{n-1}} & \cdots & -x_{0u_{n-1}}\dfrac{v_n}{v_0} + x_{nu_{n-1}} \end{pmatrix}$$

is $n-1$ at $(u,\boldsymbol{\nu}) \in C(H^T)$. It is enough to show that the rank of the matrix

$$A = \begin{pmatrix} -x_0\dfrac{v_1}{v_0} + x_1 & \cdots & -x_0\dfrac{v_n}{v_0} + x_n \\ -x_{0u_1}\dfrac{v_1}{v_0} + x_{1u_1} & \cdots & -x_{0u_1}\dfrac{v_n}{v_0} + x_{nu_1} \\ \vdots & \vdots & \vdots \\ -x_{0u_{n-1}}\dfrac{v_1}{v_0} + x_{1u_{n-1}} & \cdots & -x_{0u_{n-1}}\dfrac{v_n}{v_0} + x_{nu_{n-1}} \end{pmatrix}$$

is n at $(u,\boldsymbol{\nu}) \in C(H^T)$. We denote that $\boldsymbol{a}_i = \begin{pmatrix} x_i \\ x_{iu_1} \\ \vdots \\ x_{iu_{n-1}} \end{pmatrix}$ for $i = 0,\ldots,n$.

Then we have

$$A = \left(-\boldsymbol{a}_0\frac{v_1}{v_0} + \boldsymbol{a}_1, \ldots, -\boldsymbol{a}_0\frac{v_n}{v_0} + \boldsymbol{a}_n\right)$$

and

$$\det A = \frac{v_0}{v_0}\cdot\det(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_n) - \frac{v_1}{v_0}\cdot\det(\boldsymbol{a}_0,\boldsymbol{a}_2,\ldots,\boldsymbol{a}_n) - \cdots - \frac{v_n}{v_0}\cdot\det(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{n-1},\boldsymbol{a}_0).$$

On the other hand, we have

$$\boldsymbol{x}_+^\ell \wedge (\boldsymbol{x}_+^\ell)_{u_1} \wedge \cdots \wedge (\boldsymbol{x}_+^\ell)_{u_{n-1}} = \left(-\det(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_n), -\det(\boldsymbol{a}_0,\boldsymbol{a}_2,\ldots,\boldsymbol{a}_n), \ldots, (-1)^n\det(\boldsymbol{a}_0,\ldots,\boldsymbol{a}_{n-1})\right).$$

Since $\boldsymbol{x}_+^\ell$ is lightlike, there exists non-zero real number ξ such that $\xi\cdot\boldsymbol{x}_+^\ell(u) = (\boldsymbol{x}_+^\ell \wedge (\boldsymbol{x}_+^\ell)_{u_1} \wedge \cdots \wedge (\boldsymbol{x}_+^\ell)_{u_{n-1}})(u)$ (cf. Lemma 2.1 in [18]).

Therefore we have

$$\begin{aligned} \det A &= \left\langle \left(\frac{v_0}{v_0},\ldots,\frac{v_n}{v_0}\right), \boldsymbol{x}_+^\ell \wedge (\boldsymbol{x}_+^\ell)_{u_1} \wedge \cdots \wedge (\boldsymbol{x}_+^\ell)_{u_{n-1}} \right\rangle = \frac{1}{v_0}\left\langle \lambda\boldsymbol{x}_+^\ell + \frac{1}{4\lambda}\boldsymbol{x}_-^\ell, \xi\boldsymbol{x}_+^\ell \right\rangle \\ &= \frac{\xi}{4v_0\lambda}\left\langle \boldsymbol{x}_+^\ell, \boldsymbol{x}_-^\ell \right\rangle = -\frac{\xi}{2v_0\lambda} \neq 0 \end{aligned}$$

for $(u,\boldsymbol{\nu}) \in C(H^T)$.

Next we consider the spacelike height function. The proof is also given by direct calculations but a bit more carefully than in the previous case. We use the same notations as those of the previous case (e.g., $\boldsymbol{x}_+^\ell$ and $\boldsymbol{a}_i$ etc.). For any $\boldsymbol{\nu} \in S_1^n$, we have $-v_0^2 + v_1^2 + \cdots + v_n^2 = 1$. Without loss of the generality, we might assume that $v_n \neq 0$. We have $v_n = \pm\sqrt{1 + v_0^2 - v_1^2 - \cdots - v_{n-1}^2}$, so that

$$H^S(u,\boldsymbol{\nu}) = -x_0(u)v_0 + x_1(u)v_1 + \cdots + x_{n-1}(u)v_{n-1} \pm x_n(u)\sqrt{1 + v_0^2 - v_1^2 - \cdots - v_{n-1}^2}.$$

We also prove that the mapping

$$\Delta H^S = \left(\frac{\partial H^S}{\partial u_1}, \ldots, \frac{\partial H^S}{\partial u_{n-1}} \right)$$

is non-singular at any point. The Jacobian matrix of ΔH^S is given as follows:

$$\begin{pmatrix} \langle (\boldsymbol{x}_+^\ell)_{u_1 u_1}, \boldsymbol{\nu} \rangle & \cdots & \langle (\boldsymbol{x}_+^\ell)_{u_1 u_{n-1}}, \boldsymbol{\nu} \rangle & -x_{0u_1} + x_{nu_1} \dfrac{v_0}{v_n} & \cdots & x_{n-1u_1} - x_{nu_1} \dfrac{v_{n-1}}{v_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle (\boldsymbol{x}_+^\ell)_{u_{n-1} u_1}, \boldsymbol{\nu} \rangle & \cdots & \langle (\boldsymbol{x}_+^\ell)_{u_{n-1} u_{n-1}}, \boldsymbol{\nu} \rangle & -x_{0u_{n-1}} + x_{nu_{n-1}} \dfrac{v_0}{v_n} & \cdots & x_{n-1u_{n-1}} - x_{nu_{n-1}} \dfrac{v_{n-1}}{v_n} \end{pmatrix}.$$

We will also show that the rank of the matrix

$$\widetilde{X} = \begin{pmatrix} -x_{0u_1} + x_{nu_1} \dfrac{v_0}{v_n} & x_{1u_1} - x_{nu_1} \dfrac{v_1}{v_n} & \cdots & x_{n-1u_1} - x_{nu_1} \dfrac{v_{n-1}}{v_n} \\ \vdots & \vdots & \vdots & \vdots \\ -x_{0u_{n-1}} + x_{nu_{n-1}} \dfrac{v_0}{v_n} & x_{1u_{n-1}} - x_{nu_{n-1}} \dfrac{v_1}{v_n} & \cdots & x_{n-1u_{n-1}} - x_{nu_{n-1}} \dfrac{v_{n-1}}{v_n} \end{pmatrix}$$

is $n-1$ at $(u, \boldsymbol{\nu}) \in C(H^S)$. It should be proven that the rank of the matrix

$$\widetilde{A} = \left(-\boldsymbol{a}_0 + \boldsymbol{a}_n \frac{v_0}{v_n}, \boldsymbol{a}_1 - \boldsymbol{a}_n \frac{v_1}{v_n}, \ldots, \boldsymbol{a}_{n-1} - \boldsymbol{a}_n \frac{v_{n-1}}{v_n} \right)$$

is n at $(u, \boldsymbol{\nu}) \in C(H^S)$.

Therefore we have

$$\begin{aligned} \det \widetilde{A} &= (-1)^{n-1} \left\{ \frac{v_0}{v_n} \cdot \det(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) - \frac{v_1}{v_n} \cdot \det(\boldsymbol{a}_0, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n) + \cdots + (-1)^n \frac{v_n}{v_n} \cdot \det(\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{n-1}) \right\} \\ &= (-1)^{n-1} \left\langle \left(\frac{v_0}{v_n}, \ldots, \frac{v_n}{v_n} \right), \boldsymbol{x}_+^\ell \wedge (\boldsymbol{x}_+^\ell)_{u_1} \cdots \wedge (\boldsymbol{x}_+^\ell)_{u_{n-1}} \right\rangle \\ &= \frac{(-1)^{n-1}}{v_n} \left\langle \lambda \boldsymbol{x}_+^\ell - \frac{1}{4\lambda} \boldsymbol{x}_-^\ell, \boldsymbol{x}_+^\ell \wedge (\boldsymbol{x}_+^\ell)_{u_1} \cdots \wedge (\boldsymbol{x}_+^\ell)_{u_{n-1}} \right\rangle \\ &= \frac{(-1)^{n-1}}{v_n} \left\langle \lambda \boldsymbol{x}_+^\ell - \frac{1}{4\lambda} \boldsymbol{x}_-^\ell, \xi \boldsymbol{x}_+^\ell \right\rangle = \frac{(-1)^{n-1} \xi}{2 v_n \lambda} \neq 0 \end{aligned}$$

for $(u, \boldsymbol{\nu}) \in C(H^S)$. This completes the proof of proposition. □

By the method for constructing the Lagrangian immersion germ from Morse family (cf. Appendix A), we can define a Lagrangian immersion germ whose generating family is the timelike height function or the spacelike height function of $M = \boldsymbol{x}(U)$ as follows. For a spacelike hypersurface $\boldsymbol{x}_+^\ell : U \longrightarrow LC^*$, we denote that $\boldsymbol{x}_+^\ell(u) = (x_0(u), \ldots, x_n(u))$. Define a smooth mapping

$$L(H^T) : C(H^T) \longrightarrow T^* H^n(-1)$$

by

$$L(H^T)(u, \boldsymbol{\nu}) = \left(\boldsymbol{\nu}, -x_0(u) \frac{v_1}{v_0} + x_1(u), \ldots, -x_0(u) \frac{v_n}{v_0} + x_n(u) \right),$$

where $\boldsymbol{\nu} = (v_0, \ldots, v_n) \in H^n(-1)$ and $v_0 = \pm \sqrt{v_1^2 + \cdots + v_n^2 + 1}$. Therefore we have the local coordinate $(v_1, \ldots, v_n)$. Here we have used the triviality of the cotangent bundle $T^* H^n(-1)$.

For the de Sitter space S_1^n, we consider the local coordinate $U_i = \{\boldsymbol{\nu} = (v_0, \ldots, v_n) \in S_1^n \mid v_i \neq 0\}$. Since $T^* S_1^n | U_i$ is a trivial bundle, we define a map

$$L_i(H^S) : C(H^S) \longrightarrow T^* S_1^n | U_i \ (i = 0, 1, \ldots, n)$$

by

$$L_i(H^S)(u,\boldsymbol{v}) = \left(\boldsymbol{v}, -x_0(u) + x_i(u)\frac{v_0}{v_i}, x_1(u) - x_i(u)\frac{v_1}{v_i}, \ldots, \widehat{x_i(u) - x_i(u)\frac{v_i}{v_i}}, \ldots, x_n(u) - x_i(u)\frac{v_n}{v_i}\right),$$

where $v = (v_0, \ldots, v_n) \in S^n_1$ and we denote $(x_0, \ldots, \widehat{x_i}, \ldots, x_n)$ as a point in n-dimensional space such that the i-th component x_i is removed. We can show that if $U_i \cap U_j \neq \emptyset$ for $i \neq j$, then $L_i(H^S)$ and $L_j(H^S)$ are Lagrangian equivalent which are given by the local coordinate change of S^n_1 and Lagrangian lift of it. Indeed, we denote that the local coordinate change of S^n_1 for $i < j$; $\varphi_{ij} : U_i \longrightarrow U_j$, defined by

$$\varphi_{ij}(v_0, \ldots, \widehat{v_i}, \ldots, v_n) = \left(v_0, \ldots, v_i = \sqrt{1 + v_0^2 - v_1^2 - \cdots - \widehat{v_i^2} - \cdots - v_n^2}, \ldots, \widehat{v_j}, \ldots, v_n\right),$$

and $\widetilde{\varphi}_{ij} : T^*S^n_1 \longrightarrow T^*S^n_1$ are Lagrangian lift of φ_{ij} which defined by $\widetilde{\varphi}_{ij}(\xi) = (\varphi_{ij}^{-1})^*\xi$. Then $\widetilde{\varphi}_{ij}$ are symplectic diffeomorphism germs (cf. [1]). Also we define diffeomorphism germs $\sigma_{ij} : U \times U_i \longrightarrow U \times U_j$ by $\sigma_{ij}(u, \boldsymbol{v}) = (u, \varphi_{ij}(\boldsymbol{v}))$ and $\widetilde{\sigma}_{ij} = \sigma_{ij}\big|_{C(H^S)}$, then $\widetilde{\varphi}_{ij} \circ L_i(H^S) = L_j(H^S) \circ \widetilde{\sigma}_{ij}$ and $\varphi_{ij} \circ \pi = \pi \circ \widetilde{\varphi}_{ij}$. Therefore we can define a global Lagrangian immersion, $L(H^S) : C(H^S) \longrightarrow T^*S^n_1$.

By definition, we have the following corollary of the above proposition:

Corollary 6.2. *Under the above notations, $L(H^T)$ (respectively, $L(H^S)$) is a Lagrangian immersion such that the timelike height function $H^T : U \times H^n_+(-1) \longrightarrow \mathbb{R}$ (respectively, spacelike height function $H^S : U \times S^n_1 \longrightarrow \mathbb{R}$) of $\boldsymbol{x}^\ell_+$ is a generating family of $L(H^T)$ (respectively, $L(H^S)$).*

Therefore, we have the Lagrangian immersion $L(H^T)$ (respectively, $L(H^S)$) whose caustics is the hyperbolic evolute (respectively, de Sitter evolute) of $\boldsymbol{x}^\ell_+$. We call $L(H^T)$ (respectively, $L(H^S)$) the *Lagrangian lift* of the hyperbolic evolute (respectively, de Sitter evolute) of $\boldsymbol{x}^\ell_+$.

On the other hand, we define a mapping

$$\Psi^T : \Delta_1 \times \mathbb{R}_+ \longrightarrow T^*H^n(-1)$$

by

$$\Psi^T(\boldsymbol{v}, \boldsymbol{w}, \eta) = \left(\boldsymbol{v}, \eta\left(-(v_0 + w_0)\frac{v_1}{v_0} + (v_1 + w_1), \ldots, -(v_0 + w_0)\frac{v_n}{v_0} + (v_n + w_n)\right)\right),$$

where $\boldsymbol{v} = (v_0, v_1, \ldots, v_n)$, $\boldsymbol{w}(w_0, w_1, \ldots, w_n)$. Let α be the canonical one-form on $T^*H^n(-1)$. Then we have

$$\begin{aligned}(\Psi^T)^*\alpha &= \sum_{i=1}^{n} \eta\left(-(v_0 + w_0)\frac{v_i}{v_0} + (v_i + w_i)\right)\mathrm{d}v_i \\ &= \eta\left(-w_0\sum_{i=1}^{n}\frac{v_i}{v_0}\mathrm{d}v_i + \sum_{i=1}^{n} w_i\mathrm{d}v_i\right) \\ &= \eta\langle \mathrm{d}\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_1 = \eta\theta_{11} = -\eta\theta_{12}.\end{aligned}$$

Therefore Ψ^T is a symplectic diffeomorphism. By direct calculations, we have

$$\Psi^T \circ \widetilde{\mathcal{L}}_1(u, \phi) = L(H^T)\left(u, \frac{\exp(\phi)}{2}\boldsymbol{x}^\ell_+(u) + \frac{\exp(-\phi)}{2}\boldsymbol{x}^\ell_-(u)\right).$$

By the similar arguments as the above, we have the following theorem.

Theorem 6.3. *For any spacelike hypersurface $\boldsymbol{x}^\ell_+ : U \longrightarrow LC^*$, both the timelike height function $H^T : U \times H^n(-1) \longrightarrow \mathbb{R}$ and the spacelike height function $H^S : U \times S^n_1 \longrightarrow \mathbb{R}$ are generating families of the Lagrangian embedding $\widetilde{\mathcal{L}}_1 : U \longrightarrow \Delta_1 \times \mathbb{R}_+$.*

Since $\mathcal{B}_{H^T} = HE_M \cup (-HE_M)$ (respectively, $\mathcal{B}_{H^S} = DE_M \cup (-DE_M)$), $C_h(\widetilde{\mathcal{L}}_1) = HE_M \cup (-HE_M)$ (respectively, $C_s(\widetilde{\mathcal{L}}_1) = DE_M \cup (-DE_M)$). This means that the hyperbolic caustics (respectively, de Sitter caustics) might be identified with the hyperbolic evolute (respectively, de Sitter evolute) of M.

7. Big fronts

In this section we consider a contact manifold $(\Delta_1 \times \mathbb{R}_+ \times \mathbb{R}, \mathrm{d}\zeta + \eta\theta_{12})$, where $((\boldsymbol{v}, \boldsymbol{w}), \eta, \zeta) \in \Delta_1 \times \mathbb{R}_+ \times \mathbb{R}$. For a spacelike hypersurface $\boldsymbol{x}_+^\ell : U \longrightarrow LC^*$, we have a mapping

$$\overline{\mathcal{L}}_1 : U \times \mathbb{R} \longrightarrow \Delta_1 \times \mathbb{R}_+ \times \mathbb{R}$$

defined by

$$\overline{\mathcal{L}}_1(u, \phi) = (\widetilde{\mathcal{L}}_1(u, \phi), \exp(-\phi)).$$

Since $\widetilde{\mathcal{L}}_1$ is an embedding, $\overline{\mathcal{L}}_1$ is also an embedding. Moreover, we have

$$(\overline{\mathcal{L}}_1)^*(d\zeta + \eta\theta_{12}) = d\exp(-\phi) - (\widetilde{\mathcal{L}}_1)^*(\eta\theta_{12}) = d\exp(-\phi) - d\exp(-\phi) = 0,$$

so that $\overline{\mathcal{L}}_1$ is a Legendrian embedding. We call it the *big Legendrian embedding* associated to the Legendrian family $\mathcal{L}^\phi$.

Let $\overline{\pi}_{11} : \Delta_1 \times \mathbb{R}_+ \times \mathbb{R} \longrightarrow H^n(-1) \times \mathbb{R}$, $\overline{\pi}_{12} : \Delta_1 \times \mathbb{R}_+ \times \mathbb{R} \longrightarrow S_1^n \times \mathbb{R}$ be the canonical projections. Since $\theta_{11} = -\theta_{12}$, we have $\mathrm{d}\zeta + \eta\theta_{12} = \mathrm{d}\zeta - \eta\theta_{11}$, so that $\overline{\pi}_{11}$ and $\overline{\pi}_{12}$ are the projections of Legendrian fibrations. Moreover we have the canonical projections $\pi_{h1} : H^n(-1) \times \mathbb{R} \longrightarrow H^n(-1)$, $\pi_{h2} : H^n(-1) \times \mathbb{R} \longrightarrow \mathbb{R}$, $\pi_{d1} : S_1^n \times \mathbb{R} \longrightarrow S_1^n$ and $\pi_{d2} : S_1^n \times \mathbb{R} \longrightarrow \mathbb{R}$. Since both of $\pi_{h2} \circ \overline{\pi}_{11} \circ \overline{\mathcal{L}}_1$ and $\pi_{d2} \circ \overline{\pi}_{12} \circ \overline{\mathcal{L}}_1$ are submersions, $\overline{\mathcal{L}}_1$ is a graphlike Legendrian unfoldings with respect to both the Legendrian fibrations $\overline{\pi}_{1i}$, $i = 1, 2$. For definitions and basic properties of graphlike Legendrian unfoldings, see Appendix C.

We define two families of functions

$$\overline{H}^T : U \times H^n(-1) \times \mathbb{R} \longrightarrow \mathbb{R}$$

by $\overline{H}^T(u, \boldsymbol{v}, r) = \langle \boldsymbol{x}_+^\ell(u), \boldsymbol{v} \rangle - r = H^T(u, \boldsymbol{v}) - r$ and

$$\overline{H}^S : U \times S_1^n \times \mathbb{R} \longrightarrow \mathbb{R}$$

by $\overline{H}^S(u, \boldsymbol{v}, r) = \langle \boldsymbol{x}_+^\ell(u), \boldsymbol{v} \rangle - r = H^S(u, \boldsymbol{v}) - r$. We call $\overline{H}^T$ (respectively, $\overline{H}^S$) an *extended timelike height function* (respectively, *extended spacelike height function*). We consider the mapping

$$\overline{\Psi}^T : \Delta_1 \times \mathbb{R}_+ \times \mathbb{R} \longrightarrow T^*H^n(-1) \times \mathbb{R}$$

defined by $\overline{\Psi}^T(\boldsymbol{v}, \boldsymbol{w}, \eta, \zeta) = (\Psi^T(\boldsymbol{v}, \boldsymbol{w}, \eta), \zeta)$. We might identify $T^*H^n(-1) \times \mathbb{R}$ with 1-jet space $J^1(H^n(-1), \mathbb{R})$ whose contact structure is given by $\mathrm{d}y - \alpha$, where α is the canonical one-form on $T^*H^n(-1)$ and y is the coordinate of $\mathbb{R}$. By the previous calculation, we have

$$\left(\overline{\Psi}^T\right)^* (\mathrm{d}y - \alpha) = \mathrm{d}\zeta + \eta\theta_{12},$$

so that $\overline{\Psi}^T$ is a contact diffeomorphism. By Proposition 6.1, we have the following proposition.

Proposition 7.1. *Both the extended timelike height function $\overline{H}^T : U \times H^n(-1) \times \mathbb{R} \longrightarrow \mathbb{R}$ and the extended spacelike height function $\overline{H}^S : U \times S_1^n \times \mathbb{R} \longrightarrow \mathbb{R}$ are graphlike Morse families of hypersurfaces.*

It follows from the above proposition that we have the following theorem.

Theorem 7.2. *For any spacelike hypersurface $\boldsymbol{x}_+^\ell : U \longrightarrow LC^*$, both the extended timelike height function $\overline{H}^T : U \times H^n(-1) \times \mathbb{R} \longrightarrow \mathbb{R}$ and the extended spacelike height function $\overline{H}^S : U \times S_1^n \times \mathbb{R} \longrightarrow \mathbb{R}$ are generating families of the graphlike Legendrian unfolding $\overline{\mathcal{L}}_1$.*

8. Contact with families of hyperquadrics

In [18] we have studied the contact of spacelike hypersurfaces in LC^* with parabolic hyperquadrics as applications of theory of contact due to Montaldi [21] and the theory of Legendrian singularities. Briefly speaking, we can

completely characterize the contact of spacelike hypersurfaces with parabolic hyperquadrics in terms of the Lightcone Gauss images in generic. If we consider the spacelike parallels and the evolutes instead of the Lightcone Gauss maps, we might consider the problem what kind of geometric information we can get from the singularity of the spacelike parallels or the evolutes. We now start to give a brief review of the theory of contact due to Montaldi [21]. Let X_i, Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that the *contact of* X_1 and Y_1 at y_1 is the same type as the *contact of* X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. It is clear that in the definition $\mathbb{R}^n$ could be replaced by any manifold. In his paper [21], Montaldi gives a characterization of the notion of contact by using the terminology of singularity theory.

Theorem 8.1. *Let X_i, Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. Let $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, 0)$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(0), y_i)$. Then $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent. For the definition of $\mathcal{K}$-equivalence, see* [20].

For our purpose this theorem is not sufficient. We need the theory of contact of submanifold with families of hypersurfaces. We have two kinds of theories which describe the contact with families of hypersurfaces.

Firstly we consider the one-parameter families of hypersurfaces. Let X_i $(i = 1, 2)$ be submanifolds in $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $f_i : (\mathbb{R}^n \times \mathbb{R}, (y_i, t_i)) \longrightarrow (\mathbb{R}, 0)$ be function germs such that $f_{i,t}$ are submersion germs for any $t \in (\mathbb{R}, t_i)$. Here, we define that $f_{i,t}(y) = f_i(y, t)$. We have hypersurface germs $(\mathbb{R}^n \times \mathbb{R}, (y_i, t)) \supset \mathcal{Y}(f_i) = f_i^{-1}(0)$. We say that the *parametrized contact of* X_1 and $\mathcal{Y}_1$ at (y_1, t) is the *same type* as the *parametrized contact of* X_2 and $\mathcal{Y}_2$ at (y_2, t) if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}, (y_1, t_1)) \longrightarrow (\mathbb{R}^n \times \mathbb{R}, (y_2, t_2))$ of the form $\Phi(y, t) = (\phi(y, t), t + (t_2 - t_1))$ such that $\Phi(X_1 \times \mathbb{R}) = X_2 \times \mathbb{R}$ and $\Phi(\mathcal{Y}_1) = \mathcal{Y}_2$. In this case we write

$$PK(X_1, \mathcal{Y}_1; (y_1, t_1)) = PK(X_2, \mathcal{Y}_2; (y_2, t_2)).$$

We can show the following parametric version of Montaldi's theorem just along the line of the proof of the original theorem of Montaldi [21].

Theorem 8.2. *With the above notations, $PK(X_1, \mathcal{Y}_1; (y_1, t_1)) = PK(X_2, \mathcal{Y}_2; (y_2, t_2))$ if and only if $f_1 \circ (g_1 \times id_{\mathbb{R}})$ and $f_2 \circ (g_2 \times id_{\mathbb{R}})$ are S.P-$\mathcal{K}$-equivalent. For the definition of S.P-$\mathcal{K}$-equivalence, see Appendix* B.

Secondly we consider the codimension-1 foliation germs. Let X_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$, $g_i : (X_i, \bar{x}_i) \longrightarrow (\mathbb{R}^n, \bar{y}_i)$ be immersion germs and $f_i : (\mathbb{R}^n, \bar{y}_i) \longrightarrow (\mathbb{R}, 0)$ be submersion germs. For a submersion germ $f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$, we denote that $\mathcal{F}_f$ be the regular foliation defined by f; i.e., $\mathcal{F}_f = \{f^{-1}(c) | c \in (\mathbb{R}, 0)\}$. We say that *the contact of X_1 with the regular foliation $\mathcal{F}_{f_1}$* at $\bar{y}_1$ is the *same type* as *the contact of X_2 with the regular foliation $\mathcal{F}_{f_2}$* at $\bar{y}_2$ if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, \bar{y}_1) \longrightarrow (\mathbb{R}^n, \bar{y}_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1(c)) = Y_2(c)$, where $Y_i(c) = f_i^{-1}(c)$ for each $c \in (\mathbb{R}, 0)$. In this case we write $K(X_1, \mathcal{F}_{f_1}; \bar{y}_1) = K(X_2, \mathcal{F}_{f_2}; \bar{y}_2)$. We apply the method of Goryunov [7] to the case for $\mathcal{R}^+$-equivalences among function germs, so that we have the following:

Proposition 8.3 (*[7, Appendix]*). *Let X_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2 = n - 1$ (i.e., hypersurface), $g_i : (X_i, \bar{x}_i) \longrightarrow (\mathbb{R}^n, \bar{y}_i)$ be immersion germs and $f_i : (\mathbb{R}^n, \bar{y}_i) \longrightarrow (\mathbb{R}, 0)$ be submersion germs. We assume that $\bar{x}_i$ are singularities of function germs $f_i \circ g_i : (X_i, \bar{x}_i) \longrightarrow (\mathbb{R}, 0)$. Then $K(X_1, \mathcal{F}_{f_1}; \bar{y}_1) = K(X_2, \mathcal{F}_{f_2}; \bar{y}_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{R}^+$-equivalent. For the definition of $\mathcal{R}^+$-equivalence, see* [20].

On the other hand, Golubitsky and Guillemin [6] have given an algebraic characterization for the $\mathcal{R}^+$-equivalence among function germs. We denote $C_0^\infty(X)$ is the set of function germs $(X, 0) \longrightarrow \mathbb{R}$. Let J_f be the Jacobian ideal in $C_0^\infty(X)$ (i.e., $J_f = \langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n \rangle_{C_0^\infty(X)}$). Let $\mathcal{R}_k(f) = C_0^\infty(X)/J_f^k$ and $[f]$ be the image of f in this local ring. We say that f satisfies the *Milnor condition* if $\dim_{\mathbb{R}} \mathcal{R}_1(f) < \infty$.

Proposition 8.4 (*[6, Proposition 4.1]*). *Let f and g be germs of functions at 0 in X satisfying the Milnor condition with $df(0) = dg(0) = 0$. Then f and g are $\mathcal{R}^+$-equivalent if*

(1) *The rank and signature of the Hessians $\mathcal{H}(f)(0)$ and $\mathcal{H}(g)(0)$ are equal, and*
(2) *There is an isomorphism $\gamma : \mathcal{R}_2(f) \longrightarrow \mathcal{R}_2(g)$ such that $\gamma([f]) = [g]$.*

We now consider two families of functions

$$\mathfrak{H}^T : LC^* \times H^n(-1) \longrightarrow \mathbb{R}$$

defined by $\mathfrak{H}^T(\boldsymbol{x}, \boldsymbol{v}) = \langle \boldsymbol{x}, \boldsymbol{v} \rangle$ and

$$\mathfrak{H}^S : LC^* \times S^n_1 \longrightarrow \mathbb{R}$$

defined by $\mathfrak{H}^S(\boldsymbol{x}, \boldsymbol{v}) = \langle \boldsymbol{x}, \boldsymbol{v} \rangle$. For any $\boldsymbol{v}_0 \in H^n(-1)$, we define $\mathfrak{h}^T_{v_0}(\boldsymbol{x}) = \mathfrak{H}^T(\boldsymbol{x}, \boldsymbol{v}_0)$ and we have an elliptic hyperquadric

$$\left(\mathfrak{h}^T_{v_0}\right)^{-1}(c) = HP(\boldsymbol{v}_0, c) \cap LC^* = HL(\boldsymbol{v}_0, c).$$

By definition, $\mathfrak{h}^T_{v_0}$ is a submersion. Let $\boldsymbol{x}^\ell_+ : U \longrightarrow LC^*$ be a spacelike hypersurface. For any $u_0 \in U$, we have a timelike vector $\boldsymbol{v}_0 = (-1/2c)\boldsymbol{x}^\ell_+(u_0) + (-c/2)\boldsymbol{x}^\ell_-(u_0) \in H^n(-1)$, then we have

$$\mathfrak{h}^T_{v_0} \circ \boldsymbol{x}^\ell_+(u_0) = \mathfrak{H}^T \circ (\boldsymbol{x}^\ell_+ \times id_{H^n(-1)})(u_0, \boldsymbol{v}_0) = H^T(u_0, \boldsymbol{v}_0) = c$$

and

$$\frac{\partial(\mathfrak{h}^T_{v_0} \circ \boldsymbol{x}^\ell_+)}{\partial u_i}(u_0) = \frac{\partial H^T}{\partial u_i}(u_0, \boldsymbol{v}_0) = 0,$$

for $i = 1, \ldots, n-1$. This means that $\left(\mathfrak{h}^T_{v_0}\right)^{-1}(c) = HL(\boldsymbol{v}_0, c)$ is tangent to $M = \boldsymbol{x}^\ell_+(U)$ at $p = \boldsymbol{x}^\ell_+(u_0)$. In this case we call $HL(\boldsymbol{v}_0, c)$ a *tangent elliptic hyperquadric* of $M = \boldsymbol{x}^\ell_+(U)$ at $p = \boldsymbol{x}^\ell_+(u_0)$ with the *center* $\boldsymbol{v}_0$. We denote it as $EHL(\boldsymbol{v}_0, c)$. However, there are infinitely many tangent elliptic hyperquadrics at a general point $p = \boldsymbol{x}^\ell_+(u_0)$ depending on the real number c. If the point $\boldsymbol{v}_0$ is a point of the hyperbolic evolute of $M = \boldsymbol{x}^\ell_+(U)$, the tangent elliptic hyperquadric with the center $\boldsymbol{v}_0$ is called the *osculating elliptic hyperquadric* (or *focal elliptic hyperquadric*). Since $\mathfrak{h}^T_{v_0}$ is a submersion, we define a parallel family of elliptic hyperquadrics

$$\mathcal{EHL}(\boldsymbol{v}_0) = \left(\overline{\mathfrak{h}^T_{v_0}}\right)^{-1}(0),$$

where $\overline{\mathfrak{h}^T_{v_0}} : (LC^* \times \mathbb{R}, (\boldsymbol{v}_0, 0)) \longrightarrow (\mathbb{R}, 0)$ is defined by $\overline{\mathfrak{h}^T_{v_0}}(\boldsymbol{x}, t) = \mathfrak{h}^T_{v_0}(\boldsymbol{x}) - t$. If $\boldsymbol{v}_0 = HE^{\kappa^\ell}_M(u_0)$, then $\mathcal{EHL}(\boldsymbol{v}_0)$ is the parallel family of elliptic hyperquadrics such that the hyperquadric through $(\boldsymbol{v}_0, 0)$ is the osculating elliptic hyperquadric of $M = \boldsymbol{x}^\ell_+(U)$ with the center $\boldsymbol{v}_0$. We can also define the regular foliation

$$\mathcal{F}_{\mathfrak{h}^T_{v_0}} = \{(\mathfrak{h}^T_{v_0})^{-1}(c) \mid c \in (\mathbb{R}, 0)\}$$

whose leaves are elliptic hyperquadrics such that $(\mathfrak{h}^T_{v_0})^{-1}(0)$ is the osculating elliptic hyperquadric with the center $\boldsymbol{v}_0$. In this case $((\boldsymbol{x}^\ell_+)^{-1}(\mathcal{F}_{\mathfrak{h}^T_{v_0}}), u_0)$ is a singular foliation germ at u_0 which is called an *osculating elliptic hyperquadrical foliation* of $M = \boldsymbol{x}^\ell_+(U)$ at $p = \boldsymbol{x}^\ell_+(u_0)$. We denote it by $\mathcal{OF}^T(M, u_0)$.

Let $(\boldsymbol{x}^\ell_+)_i : (U, u_i) \longrightarrow LC^*$ $(i = \alpha, \beta)$ be spacelike hypersurface germs. We consider timelike height functions $H^T_i : (U \times H^n_+(-1), (u_i, \boldsymbol{v}_i)) \longrightarrow \mathbb{R}$ of $(\boldsymbol{x}^\ell_+)_i$, where $\boldsymbol{v}_i = HE^{\kappa^\ell_i}_{M_i}(u_i)$. We denote that $h^T_{i,v_i}(u) = H^T_i(u, \boldsymbol{v}_i)$, then we have $h^T_{i,v_i}(u) = \mathfrak{h}^T_{v_i} \circ (\boldsymbol{x}^\ell_+)_i(u)$. As an application of Appendices B and C, we have the following theorem:

Theorem 8.5. *Let $(\boldsymbol{x}^\ell_+)_i : (U, u_i) \longrightarrow LC^*$ $(i = \alpha, \beta)$ be spacelike hypersurface germs such that the corresponding graphlike Legendrian unfolding germs*

$$\mathfrak{L}_{H^T_i} : (C(H^T_i), (u_i, \boldsymbol{v}_i)) \longrightarrow J^1(H^n(-1), \mathbb{R})$$

are $S.P^+$-Legendrian stable, where $\boldsymbol{v}_i = HE^{\kappa^\ell_i}_{M_i}(u_i)$. Then the following conditions are equivalent:

(1) $PK((\boldsymbol{x}^\ell_+)_\alpha(U), \mathcal{EHL}(\boldsymbol{v}_\alpha); (\boldsymbol{x}^\ell_+)_\alpha(u_\alpha)) = PK((\boldsymbol{x}^\ell_+)_\beta(U), \mathcal{EHL}(\boldsymbol{v}_\beta); (\boldsymbol{x}^\ell_+)_\beta(u_\beta))$.
(2) $\overline{h^T_{\alpha,v_1}}$ *and* $\overline{h^T_{\beta,v_2}}$ *are $S.P$-$\mathcal{K}$-equivalent.*

(3) $\overline{H^T_\alpha}$ *and* $\overline{H^T_\beta}$ *are* $\boldsymbol{v}$-$S.P^+$-$\mathcal{K}$-*equivalent.*
(4) $\mathfrak{L}_{H^T_\alpha}$ *and* $\mathfrak{L}_{H^T_\beta}$ *are* $S.P^+$-*Legendrian equivalent.*
(5) *The graphlike unfoldings of wave fronts* $W(\mathfrak{L}_{H^T_\alpha})$ *and* $W(\mathfrak{L}_{H^T_\beta})$ *are* $S.P^+$-*diffeomorphic.*

Proof. By Theorem 8.2, the condition (1) is equivalent to the condition (2). Since both of $\mathfrak{L}_{H^T_i}$ are $S.P^+$-Legendrian stable, both of $\overline{H^T_i}$ are $S.P^+$-$\mathcal{K}$-versal deformations of $\overline{h^T_i}$ respectively (cf. Theorem B.6). By Proposition B.5, the condition (2) implies the condition (3). It always holds that the condition (3) implies the condition (2). By Theorem B.6 (1), the condition (3) is equivalent to the condition (4). Since both of $\mathfrak{L}_{H^T_i}$ are $S.P^+$-Legendrian stable, the assumption of Proposition B.8 is satisfied for $\mathfrak{L}_{H^T_i}$. It follows that the conditions (4) and (5) are equivalent. This completes the proof. □

We also have the following theorem as an application of Appendix A.

Theorem 8.6. *Let* $(\boldsymbol{x}^\ell_+)_i : (U, u_i) \longrightarrow LC^*$ $(i = \alpha, \beta)$ *be spacelike hypersurface germs such that the corresponding Lagrangian submanifold germs*

$$L(H^T_i) : (C(H^T_i), (u_i, \boldsymbol{v}_i)) \longrightarrow T^*H^n(-1)$$

are Lagrangian stable, where $\boldsymbol{v}_i = HE^{\kappa^\ell_i}_{M_i}(u_i)$. *Then the following conditions are equivalent:*

(1) $K((\boldsymbol{x}^\ell_+)_\alpha(U), \mathcal{F}_{\mathfrak{h}^T_{v_\alpha}}; (\boldsymbol{x}^\ell_+)_\alpha(u_\alpha)) = K((\boldsymbol{x}^\ell_+)_\beta(U), \mathcal{F}_{\mathfrak{h}^T_{v_\beta}}; (\boldsymbol{x}^\ell_+)_\beta(u_\beta))$.
(2) h^T_{α,v_α} *and* h^T_{β,v_β} *are* $\mathcal{R}^+$-*equivalent.*
(3) H^T_α *and* H^T_β *are* P-$\mathcal{R}^+$-*equivalent.*
(4) $L(H^T_\alpha)$ *and* $L(H^T_\beta)$ *are Lagrangian equivalent.*
(5) (a) *The rank and signature of the* $\mathcal{H}(h^T_{\alpha,v_\alpha})(u_\alpha)$ *and* $\mathcal{H}(h^T_{\beta,v_\beta})(u_\beta)$ *are equal,*
(b) *There is an isomorphism* $\gamma : \mathcal{R}_2(h^T_{\alpha,v_\alpha}) \longrightarrow \mathcal{R}_2(h^T_{\beta,v_\beta})$ *such that* $\gamma([h^T_{\alpha,v_\alpha}]) = [h^T_{\beta,v_\beta}]$.

Proof. By Proposition 8.3, the condition (1) is equivalent to the condition (2). Since both of $L(H^T_i)$ are Lagrangian stable, both of H^T_i are $\mathcal{R}^+$-versal unfoldings of h^T_{i,v_i} respectively. By the uniqueness theorem on the $\mathcal{R}^+$-versal unfolding of a function germ, the condition (2) is equivalent to the condition (3). By Theorem A.2, the condition (3) is equivalent to the condition (4). It also follows from Theorem A.2 that both of h^T_i satisfy the Milnor condition. Therefore we can apply Proposition 8.4 to our situation, so that the condition (2) is equivalent to the condition (5). This completes the proof. □

We remark that if $L(H^T_\alpha)$ and $L(H^T_\beta)$ are Lagrangian equivalent, then the corresponding hyperbolic evolutes are diffeomorphic. Since the hyperbolic evolute of a hypersurface $M = \boldsymbol{x}^\ell_+(U)$ is considered to be the caustic of $L(H^T)$, the above theorem gives a symplectic interpretation for the contact of hypersurfaces with family of hyperspheres (cf. Appendix A).

On the other hand, we have the following proposition.

Proposition 8.7. *If* $L(H^T_\alpha)$ *and* $L(H^T_\beta)$ *are Lagrangian equivalent, then the graphlike unfoldings of wave fronts* $W(\mathfrak{L}_{H^T_\alpha})$ *and* $W(\mathfrak{L}_{H^T_\beta})$ *are* $S.P^+$-*diffeomorphic.*

Proof. Since the $S.P^+$-Legendrian equivalence implies the $S.P^+$-diffeomorphism, the assertion directly follows from Proposition C.2. □

By Proposition C.3, if $\mathfrak{L}_{H^T}$ is $S.P^+$-Legendrian stable, then $L(H^T)$ is Lagrangian stable. Therefore, we have the following corollary of Theorem 8.6 and Proposition 8.7.

Corollary 8.8. *Let* $(\boldsymbol{x}^\ell_+)_i : (U, u_i) \longrightarrow LC^*$ $(i = \alpha, \beta)$ *be spacelike hypersurface germs such that the corresponding graphlike Legendrian unfolding germs*

$$\mathfrak{L}_{H^T_i} : (C(H^T_i), (u_i, \boldsymbol{v}_i)) \longrightarrow J^1(H^n(-1), \mathbb{R})$$

are $S.P^+$-*Legendrian stable, where* $\boldsymbol{v}_i = HE^{\kappa^\ell_i}_{M_i}(u_i)$.

If $K((\boldsymbol{x}^\ell_+)_\alpha(U), \mathcal{F}_{\mathfrak{h}^T_{v_\alpha}}; (\boldsymbol{x}^\ell_+)_\alpha(u_\alpha)) = K((\boldsymbol{x}^\ell_+)_\beta(U), \mathcal{F}_{\mathfrak{h}^T_{v_\beta}}; (\boldsymbol{x}^\ell_+)_\beta(u_\beta))$*, then the graphlike unfoldings of wave fronts* $W(\mathfrak{L}_{H^T_\alpha})$ *and* $W(\mathfrak{L}_{H^T_\beta})$ *are* $S.P^+$*-diffeomorphic.*

Proof. By Theorem 8.6, if $K((\boldsymbol{x}^\ell_+)_\alpha(U), \mathcal{F}_{\mathfrak{h}^T_{v_\alpha}}; (\boldsymbol{x}^\ell_+)_\alpha(u_\alpha)) = K((\boldsymbol{x}^\ell_+)_\beta(U), \mathcal{F}_{\mathfrak{h}^T_{v_\beta}}; (\boldsymbol{x}^\ell_+)_\beta(u_\beta))$, then $L(H^T_\alpha)$ and $L(H^T_\beta)$ are Lagrangian equivalent, so that $\mathfrak{L}_{H^T_\alpha}$ and $\mathfrak{L}_{H^T_\beta}$ are SP^+-Legendrian equivalent by Proposition 8.7. □

Corollary 8.9. *Under the same assumptions as those of Theorem 8.6, we have the following: If one of the conditions of Theorem 8.6 is satisfied then*

(1) *The hyperbolic evolutes* HE_{M_α} *and* HE_{M_β} *are diffeomorphic as germs.*
(2) *The osculating elliptic hyperquadrical foliation germs* $\mathcal{OF}^T(M_\alpha, u_\alpha)$ *and* $\mathcal{OF}^T(M_\beta, u_\beta)$ *are diffeomorphic as germs.*

Similarly we can construct the *osculating hyperbolic hyperquadric* (or *focal hyperbolic hyperquadric*) of a spacelike hypersurface $\boldsymbol{x}^\ell_+ : U \longrightarrow LC^*$ by using a function $\mathfrak{H}^S : LC^* \times S^n_1 \longrightarrow \mathbb{R}$. For any $\boldsymbol{v}_0 \in S^n_1$, we also denote that $\mathfrak{h}^S_{v_0}(\boldsymbol{x}) = \mathfrak{H}^S(\boldsymbol{x}, \boldsymbol{v}_0)$ and we have $h^S_{v_0}(u) = \mathfrak{h}^S_{v_0} \circ \boldsymbol{x}^\ell_+(u)$. We can show that $\left(\mathfrak{h}^S_{v_0}\right)^{-1}(c) = HL(\boldsymbol{v}_0, c)$ is tangent to $M = \boldsymbol{x}^\ell_+(U)$ at $p = \boldsymbol{x}^\ell_+(u_0)$. In this case we call $HL(\boldsymbol{v}_0, c)$ a *tangent hyperbolic hyperquadric* of $M = \boldsymbol{x}^\ell_+(U)$ at $p = \boldsymbol{x}^\ell_+(u_0)$ with the *center* $\boldsymbol{v}_0$, we denote it $HHL(\boldsymbol{v}_0, c)$. However, there are infinitely many tangent hyperbolic hyperquadrics at a general point $p = \boldsymbol{x}^\ell_+(u_0)$ depending on the real number c. If the point $\boldsymbol{v}_0$ is a point of the de Sitter evolute of $M = \boldsymbol{x}^\ell_+(U)$, the tangent hyperbolic hyperquadric with the center $\boldsymbol{v}_0$ is called the *osculating hyperbolic hyperquadric* (or *focal hyperbolic hyperquadric*). Since $\mathfrak{h}^S_{v_0}$ is a submersion, we define a parallel family of hyperbolic hyperquadrics

$$\mathcal{HHL}(\boldsymbol{v}_0) = \left(\overline{\mathfrak{h}^S_{v_0}}\right)^{-1}(0),$$

where $\overline{\mathfrak{h}^S_{v_0}} : (LC^* \times \mathbb{R}, (\boldsymbol{v}_0, 0)) \longrightarrow (\mathbb{R}, 0)$ is defined by $\overline{\mathfrak{h}^S_{v_0}}(\boldsymbol{x}, t) = \mathfrak{h}^S_{v_0}(\boldsymbol{x}) - t$. If $\boldsymbol{v}_0 = DE^{\kappa^\ell}_M(u_0)$, then $\mathcal{HHL}(\boldsymbol{v}_0)$ is the parallel family of hyperbolic hyperquadrics such that the hyperquadric through $(\boldsymbol{v}_0, 0)$ is the osculating hyperbolic hyperquadric of $M = \boldsymbol{x}^\ell_+(U)$ with the center $\boldsymbol{v}_0$. We can also define the regular foliation

$$\mathcal{F}_{\mathfrak{h}^S_{v_0}} = \{(\mathfrak{h}^S_{v_0})^{-1}(c) \mid c \in (\mathbb{R}, 0)\}$$

whose leaves are hyperbolic hyperquadrics such that $(\mathfrak{h}^S_{v_0})^{-1}(0)$ is the osculating hyperbolic hyperquadric with the center $\boldsymbol{v}_0$. In this case $((\boldsymbol{x}^\ell_+)^{-1}(\mathcal{F}_{\mathfrak{h}^S_{v_0}}), u_0)$ is a singular foliation germ at u_0 which is called an *osculating hyperbolic hyperquadrical foliation* of $M = \boldsymbol{x}^\ell_+(U)$ at $p = \boldsymbol{x}^\ell_+(u_0)$. We denote it by $\mathcal{OF}^S(M, u_0)$. Then we have the following theorems:

Theorem 8.10. *Let* $(\boldsymbol{x}^\ell_+)_i : (U, u_i) \longrightarrow LC^*$ $(i = \alpha, \beta)$ *be spacelike hypersurface germs such that the corresponding graphlike Legendrian unfolding germs*

$$\mathfrak{L}_{H^S_i} : (C(H^S_i), (u_i, \boldsymbol{v}_i)) \longrightarrow J^1(S^n_1, \mathbb{R})$$

are $S.P^+$*-Legendrian stable, where* $\boldsymbol{v}_i = DE^{\kappa^\ell_i}_{M_i}(u_i)$*. Then the following conditions are equivalent:*

(1) $PK((\boldsymbol{x}^\ell_+)_\alpha(U), \mathcal{HHL}(\boldsymbol{v}_\alpha); (\boldsymbol{x}^\ell_+)_\alpha(u_\alpha)) = PK((\boldsymbol{x}^\ell_+)_\beta(U), \mathcal{HHL}(\boldsymbol{v}_\beta); (\boldsymbol{x}^\ell_+)_\beta(u_\beta))$.
(2) $\overline{h^S_{\alpha,v_1}}$ *and* $\overline{h^S_{\beta,v_2}}$ *are* $S.P$*-*$\mathcal{K}$*-equivalent.*
(3) $\overline{H^S_\alpha}$ *and* $\overline{H^S_\beta}$ *are* $\boldsymbol{v}$*-*$S.P^+\mathcal{K}$*-equivalent.*
(4) $\mathfrak{L}_{H^S_\alpha}$ *and* $\mathfrak{L}_{H^S_\beta}$ *are* $S.P^+$*-Legendrian equivalent.*
(5) *The graphlike unfoldings of wave fronts* $W(\mathfrak{L}_{H^S_\alpha})$ *and* $W(\mathfrak{L}_{H^S_\beta})$ *are* $S.P^+$*-diffeomorphic.*

Theorem 8.11. *Let $(\boldsymbol{x}^{\ell}_{+})_i : (U, u_i) \longrightarrow LC^*$ $(i = \alpha, \beta)$ be spacelike hypersurface germs such that the corresponding Lagrangian submanifold germs*

$$L(H_i^S) : (C(H_i^S), (u_i, \boldsymbol{v}_i)) \longrightarrow T^*S^n_1$$

are Lagrangian stable, where $\boldsymbol{v}_i = DE^{\kappa_i^{\ell}}_{M_i}(u_i)$. Then the following conditions are equivalent:

(1) $K((\boldsymbol{x}^{\ell}_{+})_\alpha(U), \mathcal{F}_{\mathfrak{h}^S_{v_\alpha}}; (\boldsymbol{x}^{\ell}_{+})_\alpha(u_\alpha)) = K((\boldsymbol{x}^{\ell}_{+})_\beta(U), \mathcal{F}_{\mathfrak{h}^S_{v_\beta}}; (\boldsymbol{x}^{\ell}_{+})_\beta(u_\beta))$.

(2) h^S_{α,v_α} *and* h^S_{β,v_β} *are* $\mathcal{R}^+$*-equivalent.*

(3) H^S_α *and* H^S_β *are* P*-*$\mathcal{R}^+$*-equivalent.*

(4) $L(H^S_\alpha)$ *and* $L(H^S_\beta)$ *are Lagrangian equivalent.*

(5) (a) *The rank and signature of the* $\mathcal{H}(h^S_{\alpha,v_\alpha})(u_\alpha)$ *and* $\mathcal{H}(h^S_{\beta,v_\beta})(u_\beta)$ *are equal,*

(b) *There is an isomorphism* $\gamma : \mathcal{R}_2(h^S_{\alpha,v_\alpha}) \longrightarrow \mathcal{R}_2(h^S_{\beta,v_\beta})$ *such that* $\gamma([h^S_{\alpha,v_\alpha}]) = [h^S_{\beta,v_\beta}]$.

The proofs of the above theorems are direct analogies of the corresponding proofs of Theorems 8.5 and 8.6, so that we omit the proofs.

We also have the following proposition. Since the proofs are also direct analogies of the proofs of Proposition 8.7 and Corollary 8.8, we omit them.

Proposition 8.12. *If* $L(H^S_\alpha)$ *and* $L(H^S_\beta)$ *are Lagrangian equivalent, then the graphlike unfoldings of wave fronts* $W(\mathfrak{L}_{H^S_\alpha})$ *and* $W(\mathfrak{L}_{H^S_\beta})$ *are* $S.P^+$*-diffeomorphic.*

By Proposition C.3, if $\mathfrak{L}_{H^S}$ is $S.P^+$-Legendrian stable, then $L(H^S)$ is Lagrangian stable. Therefore, we also have the following corollary.

Corollary 8.13. *Let* $(\boldsymbol{x}^{\ell}_{+})_i : (U, u_i) \longrightarrow LC^*$ $(i = \alpha, \beta)$ *be spacelike hypersurface germs such that the corresponding graphlike Legendrian unfolding germs*

$$\mathfrak{L}_{H_i^S} : (C(H_i^S), (u_i, \boldsymbol{v}_i)) \longrightarrow J^1(S^n_1, \mathbb{R})$$

are $S.P^+$*-Legendrian stable, where* $\boldsymbol{v}_i = DE^{\kappa_i^{\ell}}_{M_i}(u_i)$.

If $K((\boldsymbol{x}^{\ell}_{+})_\alpha(U), \mathcal{F}_{\mathfrak{h}^S_{v_\alpha}}; (\boldsymbol{x}^{\ell}_{+})_\alpha(u_\alpha)) = K((\boldsymbol{x}^{\ell}_{+})_\beta(U), \mathcal{F}_{\mathfrak{h}^S_{v_\beta}}; (\boldsymbol{x}^{\ell}_{+})_\beta(u_\beta))$, *then the graphlike unfoldings of wave fronts* $W(\mathfrak{L}_{H^S_\alpha})$ *and* $W(\mathfrak{L}_{H^S_\beta})$ *are* $S.P^+$*-diffeomorphic.*

Corollary 8.14. *Under the same assumptions as those of Theorem 8.11, we have the following. If one of the conditions of Theorem 8.11 is satisfied then*

(1) *The de Sitter evolutes* DE_{M_α} *and* DE_{M_β} *are diffeomorphic as germs.*

(2) *The osculating hyperbolic hyperquadrical foliation germs* $\mathcal{OF}^S(M_\alpha, u_\alpha)$ *and* $\mathcal{OF}^S(M_\beta, u_\beta)$ *are diffeomorphic as germs.*

Remark. If we assume that $\boldsymbol{x}^h : U \longrightarrow H^n(-1)$ is an embedding, we can get the information of the contact with families of hyperspheres or equidistant hypersurfaces in $H^n(-1)$. Analogous assertion to Theorem 8.6 was given as Theorem 5.3 in [14]. Moreover, if we consider a spacelike embedding $\boldsymbol{x}^d : U \longrightarrow S^n_1$, we also get the information of the contact with families of hyperbolic hyperquadrics or elliptic hyperquadrics in S^n_1. However the arguments are almost the same as the previous case, so we omit the details.

9. Generic properties

In this section we consider generic properties of spacelike hypersurfaces in pseudo-spheres. The main tool is a kind of transversality theorems. We consider the space of spacelike embeddings $\mathrm{Emb}_s(U, LC^*)$ with Whitney C^∞-topology. We also consider the functions $\mathfrak{H}^T : LC^* \times H^n(-1) \longrightarrow \mathbb{R}$ and $\mathfrak{H}^S : LC^* \times S^n_1 \longrightarrow \mathbb{R}$ which have been

defined in Section 8. We claim that $\mathfrak{h}_v^T$ (respectively, $\mathfrak{h}_v^S$) is a submersion for any $\boldsymbol{v} \in H^n(-1)$ (respectively, $\boldsymbol{v} \in S_1^n$). For any $\boldsymbol{x}_+^\ell \in \mathrm{Emb}_s(U, LC^*)$, we have $H^T = \mathfrak{H}^T \circ (\boldsymbol{x}_+^\ell \times id_{H^n(-1)})$ (respectively, $H^S = \mathfrak{H}^S \circ (\boldsymbol{x}_+^\ell \times id_{S_1^n})$). We also have the r-jet extension of $\overline{H^T}$ (respectively, $\overline{H^S}$):

$$j_1^r\overline{H^T} : U \times H^n(-1) \times \mathbb{R} \longrightarrow J^r(U \times \mathbb{R}, \mathbb{R}) \quad \text{(respectively, } j_1^r\overline{H^S} : U \times S_1^n \times \mathbb{R} \longrightarrow J^r(U \times \mathbb{R}, \mathbb{R})\text{)}$$

defined by $j_1^r\overline{H^T}(u, \boldsymbol{v}, t) = j^r\overline{h_v^T}(u, t)$ (respectively, $j_1^r\overline{H^S}(u, \boldsymbol{v}, t) = j^r\overline{h_v^S}(u, t)$). We consider the trivialization $J^r(U \times \mathbb{R}, \mathbb{R}) = (U \times \mathbb{R}) \times \mathbb{R} \times J^r((n-1)+1, 1)$. For any submanifold $Q \subset J^r((n-1)+1, 1)$, we denote that $\widetilde{Q} = (U \times \mathbb{R}) \times \{0\} \times Q$. Then we have the following proposition as a corollary of Lemma 6 in Wassermann [26]. (See also Montaldi [22].)

Proposition 9.1. *Let Q be a submanifold of $J^r((n-1)+1, 1)$. Then the set*

$$T_Q^X = \{\boldsymbol{x}_+^\ell \in \mathrm{Emb}_s(U, LC^*) \mid j_1^\ell\overline{H^X} \text{ is transversal to } \widetilde{Q}\}$$

is a residual subset of $\mathrm{Emb}_s(U, LC^*)$, where $X = T, S$. If Q is a closed subset, then T_Q^X is open.

For $n \leq 4$, we have a finite list of a generic classification of function germs $f : (\mathbb{R}^k \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ by the $S.P$-$\mathcal{K}$-equivalence (cf. Zakalyukin [28] or Izumiya [10, Theorem 4.2]). By the above proposition and Proposition B.7, we have the following theorem.

Theorem 9.2. *Assume that $n \leq 4$ and $X = T, S$. There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}_s(U, LC^*)$ such that for any $\boldsymbol{x}_+^\ell \in \mathcal{O}$, the germ of the graphlike Legendrian unfolding $\mathfrak{L}_{H^X}$ at each point is $S.P^+$-Legendrian stable.*

Remark. If we consider the space of embeddings into hyperbolic space $\mathrm{Emb}(U, H^n(-1))$ or the space of spacelike embeddings into de Sitter space $\mathrm{Emb}_s(U, S_1^n)$, we have the similar results as the above assertions. Moreover, we have the universal definitions of spacelike parallels and evolutes in pseudo-spheres, so that the generic classifications are the same as those of the above case.

10. The cases $n = 3$

In this section we consider the case $n = 3$. By Theorem 9.2, there exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}_s(U, LC^*)$ such that for any $\boldsymbol{x}_+^\ell \in \mathcal{O}$, the germ of the graphlike Legendrian unfolding $\mathfrak{L}_{H^X}$ at each point is $S.P^+$-Legendrian stable, where $X = T, S$. By the local classification theorem on graphlike Legendrian unfoldings by the $S.P^+$-Legendrian equivalence [10,28], the corresponding graphlike Legendrian unfolding germ $\mathfrak{L}_{H^X}$ $(X = T, S)$ at any point is $S.P^+$-Legendrian equivalent to a graphlike Legendrian unfoldings whose graphlike generating family is stably x-$S.P^+$-$\mathcal{K}$-equivalent to one of the following germs:

$$\begin{aligned}
&A_2 : q_1^3 + x_1q_1 + x_2 + x_3 + t,\\
&A_3^\pm : q_1^4 + x_1q_1^2 + x_2q_1 + x_3 \pm t,\\
&A_4 : q_1^5 + x_1q_1^3 + x_2q_1^2 + x_3q_1 + t,\\
&D_4^\pm : q_2^2q_1 \pm q_1^3 + x_1q_1^2 + x_2q_1 + x_3q_2 + t.
\end{aligned}$$

We remark that the germs of types $B_2, B_3, C_3, B_4, C_4, F_4$ appeared in the classification of big fronts in [28]. However, these germs cannot be realized as graphlike generating families.

On the other hand, by Proposition C.3, the corresponding Lagrangian submanifold germ $L(H^X)(C(H^X))$ at any point is Lagrangian stable for any $\boldsymbol{x}_+^\ell \in \mathcal{O}$. By definition (cf. Appendix C), $S.P^+$-Legendrian equivalence among graphlike Legendrian unfoldings preserves both the caustics and the perestroikas of wavefronts up to local diffeomorphism. This equivalence relation clarifies the "local differential topology" of both the caustics and the perestroikas of wavefronts. On the other hand, by Proposition C.2, the Lagrangian equivalence among Lagrangian submanifold germs is a stronger equivalence relation than the $S.P^+$-Legendrian equivalence among corresponding graphlike Legendrian unfoldings. Therefore, it is enough to consider the Lagrangian equivalence for low dimensional case such as the case $n = 3$. By the classification theorem of stable Lagrangian submanifold (cf., [1], Page 330,

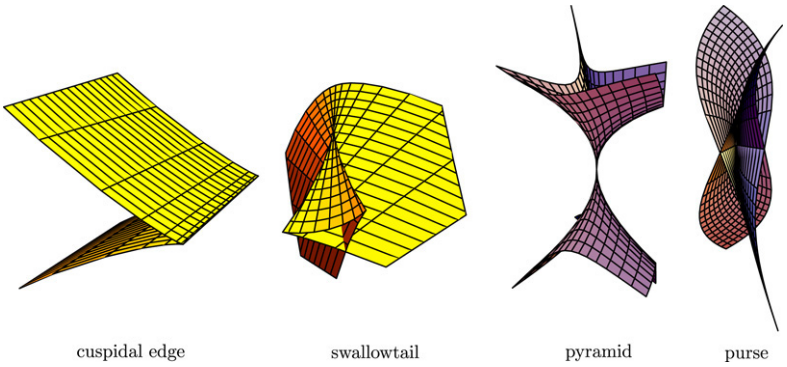

Fig. 1. Generic hyperbolic (resp. de Sitter) evolute germs ($n = 3$).

Corollary 2), the corresponding Lagrangian submanifold germ $L(H^X)(C(H^X))$ at any point is Lagrangian equivalent to a Lagrangian submanifold germ whose generating family is stably P-$\mathcal{R}^+$-equivalent to one of the following germs:

$$A_2 : q_1^3 + x_1 q_1 + x_2 + x_3,$$
$$A_3^{\pm} : \pm q_1^4 + x_1 q_1^2 + x_2 q_1 + x_3,$$
$$A_4 : q_1^5 + x_1 q_1^3 + x_2 q_1^2 + x_3 q_1,$$
$$D_4^{\pm} : q_2^2 q_1 \pm q_1^3 + x_1 q_1^2 + x_2 q_1 + x_3 q_2.$$

Since the total evolute is the caustics of the Lagrangian submanifold $\widetilde{\mathcal{L}}_1(U)$ in $\Delta_1 \times \mathbb{R}_+$ whose generating families are H^T and H^S, we have the following theorem as an application of the above classification and Corollaries 8.9 and 8.14.

Theorem 10.1. *For any $\boldsymbol{x}_+^\ell \in \mathcal{O}$ and any point $(u_0, \boldsymbol{v}_0) \in U \times H^3(-1)$ (respectively, $(u_0, \boldsymbol{v}_0) \in U \times S_1^3$), we have the following assertions:*

(1) *The hyperbolic evolute germ $(HE_M, \boldsymbol{v}_0)$ (respectively, de Sitter evolute germ $(DE_M, \boldsymbol{v}_0)$) is diffeomorphic to the fold (A_2), the cuspidal edge $(A_3^{\pm})$, the swallowtail (A_4), the pyramid (D_4^-) or the purse (D_4^+).*

(2) *The osculating elliptic (respectively, hyperbolic) hyperquadrical foliation germ $\mathcal{OF}^T(M, u_0)$ (respectively, $\mathcal{OF}^S(M, u_0)$) is diffeomorphic to the foliation germs $(\mathcal{F}_f, 0)$ with $f(q_1, q_2) = F(q_1, q_2, \boldsymbol{0})$, where $F(q_1, q_2, x_1, x_2, x_3)$ is one of the germs of type $A_2, A_3^{\pm}, A_4, D_4^{\pm}$.*

Here, the pictures of the cuspidal edge, the swallowtail, the pyramid and the purse are given in Fig. 1.
We can also draw the pictures of the foliation germs $\mathcal{F}_f$ in Theorem 10.1, see Fig. 2.

Acknowledgements

The first author's work was partially supported by Grant-in-Aid for formation of COE. 'Mathematics of Nonlinear Structure via Singularities'. The second author's work was partially supported by JSPS Research Fellowships for Young Scientists.

Appendix A. The theory of Lagrangian singularities

In this section we give a brief review on the theory of Lagrangian singularities due to [1]. We consider the cotangent bundle $\pi : T^*\mathbb{R}^n \longrightarrow \mathbb{R}^n$ over $\mathbb{R}^n$. Let $(x, p) = (x_1, \ldots, x_n, p_1, \ldots, p_n)$ be the canonical coordinate on $T^*\mathbb{R}^n$. Then the canonical symplectic structure on $T^*\mathbb{R}^n$ is given by the *canonical two form* $\omega = \sum_{i=1}^n \mathrm{d}p_i \wedge \mathrm{d}x_i$. Let $i : L \longrightarrow T^*\mathbb{R}^n$ be an immersion. We say that i is a *Lagrangian immersion* if $\dim L = n$ and $i^*\omega = 0$. In this case the critical value of $\pi \circ i$ is called the *caustic* of $i : L \longrightarrow T^*\mathbb{R}^n$ and it is denoted by C_L. The main result in the

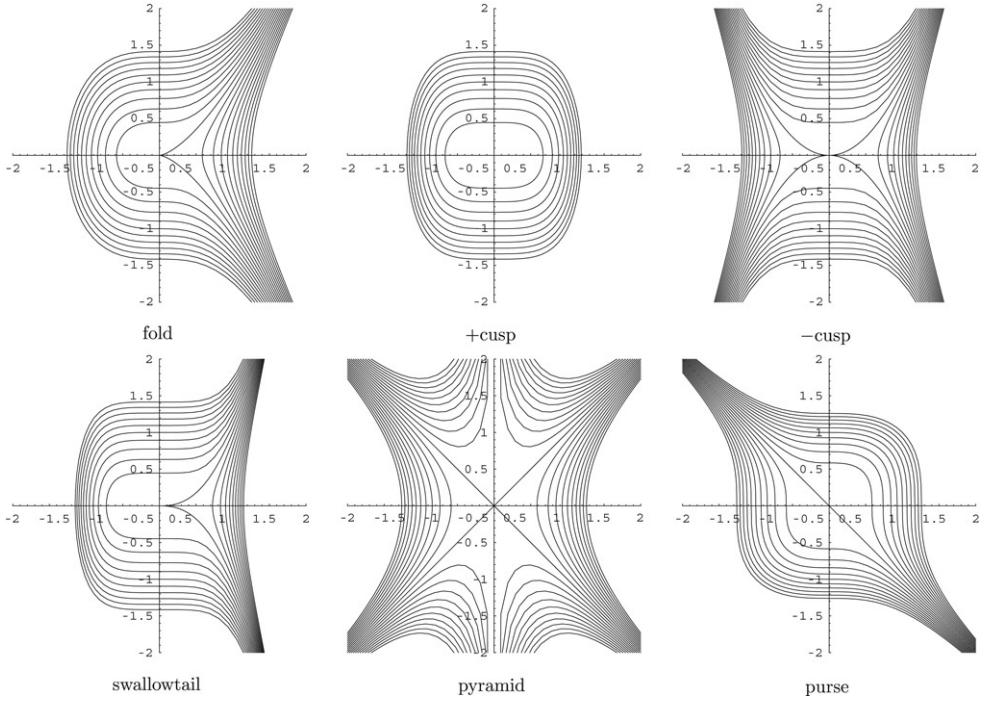

Fig. 2. Generic osculating elliptic (resp. hyperbolic) hyperquadrical foliation germs ($n = 3$).

theory of Lagrangian singularities is to describe Lagrangian immersion germs by using families of function germs. Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ be an n-parameter unfolding of function germs. We call

$$C(F) = \left\{ (q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, 0) \,\middle|\, \frac{\partial F}{\partial q_1}(q, x) = \cdots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\},$$

the *catastrophe set* of F and

$$\mathcal{B}_F = \left\{ x \in (\mathbb{R}^n, 0) \,\middle|\, \exists (q, x) \in C(F) \text{ such that rank } \left(\frac{\partial^2 F}{\partial q_i \partial q_j}(q, x) \right) < k \right\}$$

the *bifurcation set* of F.

Let $\pi_n : (\mathbb{R}^k \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ be the canonical projection, then we can easily show that the bifurcation set of F is the critical value set of $\pi_n|_{C(F)}$. We say that F is a *Morse family of functions* if the map germ

$$\Delta F = \left(\frac{\partial F}{\partial q_1}, \ldots, \frac{\partial F}{\partial q_k} \right) : (\mathbb{R}^k \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^k, 0)$$

is non-singular, where $(q, x) = (q_1, \ldots, q_k, x_1, \ldots, x_n) \in (\mathbb{R}^k \times \mathbb{R}^n, 0)$. In this case we have a smooth submanifold germ $C(F) \subset (\mathbb{R}^k \times \mathbb{R}^n, 0)$ and a map germ $L(F) : (C(F), 0) \longrightarrow T^*\mathbb{R}^n$ defined by

$$L(F)(q, x) = \left(x, \frac{\partial F}{\partial x_1}(q, x), \ldots, \frac{\partial F}{\partial x_n}(q, x) \right).$$

We can show that $L(F)$ is a Lagrangian immersion. Then we have the following fundamental theorem ([1], page 300).

Proposition A.1. *All Lagrangian submanifold germs in $T^*\mathbb{R}^n$ are constructed by the above method.*

Under the above notation, we call F a *generating family* of $L(F)$.

We define an equivalence relation among Lagrangian immersion germs. Let $i : (L, x) \longrightarrow (T^*\mathbb{R}^n, p)$ and $i' : (L', x') \longrightarrow (T^*\mathbb{R}^n, p')$ be Lagrangian immersion germs. Then we say that i and i' are *Lagrangian equivalent* if

there exists a diffeomorphism germ $\sigma : (L, x) \longrightarrow (L', x')$, a symplectic diffeomorphism germ $\tau : (T^*\mathbb{R}^n, p) \longrightarrow (T^*\mathbb{R}^n, p')$ and a diffeomorphism germ $\bar{\tau} : (\mathbb{R}^n, \pi(p)) \longrightarrow (\mathbb{R}^n, \pi(p'))$ such that $\tau \circ i = i' \circ \sigma$ and $\pi \circ \tau = \bar{\tau} \circ \pi$, where $\pi : (T^*\mathbb{R}^n, p) \longrightarrow (\mathbb{R}^n, \pi(p))$ is the canonical projection and a symplectic diffeomorphism germ is a diffeomorphism germ which preserves symplectic structure on $T^*\mathbb{R}^n$. In this case the caustic C_L is diffeomorphic to the caustic $C_{L'}$ by the diffeomorphism germ $\bar{\tau}$.

A Lagrangian immersion germ into $T^*\mathbb{R}^n$ at a point is said to be *Lagrangian stable* if for every map with the given germ there is a neighborhood in the space of Lagrangian immersions (in the Whitney C^∞-topology) and a neighborhood of the original point such that each Lagrangian immersion belonging to the first neighborhood has in the second neighborhood a point at which its germ is Lagrangian equivalent to the original germ.

We can interpret the Lagrangian equivalence by using the notion of generating families. Let $\mathcal{E}_x$ be the ring of function germs of $x = (x_1, \ldots, x_n)$ variables at the origin and $\mathfrak{M}_x = \{h \in \mathcal{E}_x \mid h(0) = 0\}$ be the unique maximal ideal. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ be function germs. We say that F and G are *P-$\mathcal{R}^+$-equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^k \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n, 0)$ of the form $\Phi(q, x) = (\Phi_1(q, x), \phi(x))$ and a function germ $h : (\mathbb{R}^n, 0) \longrightarrow \mathbb{R}$ such that $G(q, x) = F(\Phi(q, x)) + h(x)$. For any $F_1 : (\mathbb{R}^k \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ and $F_2 : (\mathbb{R}^{k'} \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$, F_1 and F_2 are said to be *stably P-$\mathcal{R}^+$-equivalent* if they become P-$\mathcal{R}^+$-equivalent after the addition to the arguments to q_i of new arguments q_i' and to the functions F_i of nondegenerate quadratic forms Q_i in the new arguments (i.e., $F_1 + Q_1$ and $F_2 + Q_2$ are P-$\mathcal{R}^+$-equivalent).

Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ be a function germ. We say that F is an *$\mathcal{R}^+$-versal deformation* of $f = F|_{\mathbb{R}^k \times \{0\}}$ if

$$\mathcal{E}_q = J_f + \left\langle \frac{\partial F}{\partial x_1}\middle| \mathbb{R}^k \times \{0\}, \ldots, \frac{\partial F}{\partial x_n}\middle| \mathbb{R}^k \times \{0\} \right\rangle_{\mathbb{R}} + \langle 1 \rangle_{\mathbb{R}},$$

where

$$J_f = \left\langle \frac{\partial f}{\partial q_1}(q), \ldots, \frac{\partial f}{\partial q_k}(q) \right\rangle_{\mathcal{E}_q}.$$

Theorem A.2. *Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ and $G : (\mathbb{R}^{k'} \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ be Morse families of functions. Then we have the following:*

(1) *$L(F)$ and $L(G)$ are Lagrangian equivalent if and only if F and G are stably P-$\mathcal{R}^+$-equivalent.*
(2) *$L(F)$ is a Lagrangian stable if and only if F is a $\mathcal{R}^+$-versal deformation of $F|\mathbb{R}^k \times \{0\}$.*

For the proof of the above theorem, see ([1], pages 304 and 325). The following proposition describes the well-known relationship between bifurcation sets and equivalence among unfoldings of function germs:

Proposition A.3. *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ be function germs. If F and G are P-$\mathcal{R}^+$-equivalent then there exist a diffeomorphism germ $\phi : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ such that $\phi(\mathcal{B}_F) = \mathcal{B}_G$.*

Appendix B. Families of wave fronts and discriminant

In this appendix we give a brief review of the classification theory of both the families of wave fronts and the discriminants. Almost all results are given by Zakalyukin [28]. However, we give some detailed information here which might be new. Moreover some equivalence relations presented here have been independently introduced by the first named author [10] for different purposes from those of Zakalyukin [28].

We consider the projective cotangent bundle $\pi : PT^*(\mathbb{R}^n \times \mathbb{R}) \longrightarrow \mathbb{R}^n \times \mathbb{R}$ over $\mathbb{R}^n \times \mathbb{R}$. Let $\Pi : TPT^*(\mathbb{R}^n \times \mathbb{R}) \longrightarrow PT^*(\mathbb{R}^n \times \mathbb{R})$ be the tangent bundle over $PT^*(\mathbb{R}^n \times \mathbb{R})$ and $\mathrm{d}\pi : TPT^*(\mathbb{R}^n \times \mathbb{R}) \longrightarrow T(\mathbb{R}^n \times \mathbb{R})$ the differential map of π.

For any $X \in TPT^*(\mathbb{R}^n \times \mathbb{R})$, there exists an element $\alpha \in T^*_{(x,y)}(\mathbb{R}^n \times \mathbb{R})$ such that $\Pi(X) = [\alpha]$. For an element $V \in T_{(x,y)}(\mathbb{R}^n \times \mathbb{R})$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define *the canonical contact structure* on $PT^*(\mathbb{R}^n \times \mathbb{R})$ by

$$K = \{X \in TPT^*(\mathbb{R}^n \times \mathbb{R}) | \Pi(X)(\mathrm{d}\pi(X)) = 0\}.$$

Because of the trivialization $PT^*(\mathbb{R}^n \times \mathbb{R}) \cong (\mathbb{R}^n \times \mathbb{R}) \times P(\mathbb{R}^n \times \mathbb{R})^*$, we call

$$((x_1, \ldots, x_n, t), [\xi_1 : \cdots : \xi_n : \tau])$$

a homogeneous coordinate, where $[\xi_1 : \cdots : \xi_n : \tau]$ is the homogeneous coordinate of the dual projective space $P(\mathbb{R}^n \times \mathbb{R})^*$. It is easy to show that $X \in K_{((x,y),[\xi:\tau])}$ if and only if $\sum_{i=1}^n \mu_i \xi_i + \lambda\tau = 0$, where $\mathrm{d}\pi(X) = \sum_{i=1}^n \mu_i \frac{\partial}{\partial x_i} + \lambda \frac{\partial}{\partial t}$.

We remark that $PT^*(\mathbb{R}^n \times \mathbb{R})$ is a fibrewise compactification of the 1-jet space $J^1(\mathbb{R}^n, \mathbb{R})$ as follows. We consider an affine open subset $U_\tau = \{((x,t),[\xi : \tau]) | \tau \neq 0\}$ of $PT^*(\mathbb{R}^n \times \mathbb{R})$. For any $((x,t),[\xi:\tau]) \in U_\tau$, we have

$$((x_1, \ldots, x_n, t), [\xi_1 : \cdots : \xi_n : \tau]) = \left((x_1, \ldots, x_n, t), \left[-\frac{\xi_1}{\tau} : \cdots : -\frac{\xi_n}{\tau} : -1\right]\right),$$

so that we may adopt the corresponding *affine coordinates* $((x_1, \ldots, x_n, t), (p_1, \ldots, p_n))$, where $p_i = -\xi_i/\tau$. On U_τ we can easily show that $\theta^{-1}(0) = K|U_\tau$, where $\theta = \mathrm{d}t - \sum_{i=1}^n p_i \mathrm{d}x_i$. This means that U_τ may be identified with the 1-jet space $J^1(\mathbb{R}^n, \mathbb{R})$. We call the above coordinate *a system of canonical coordinates.* Throughout the remainder of this paper, we use this identification so that we have $J^1(\mathbb{R}^n, \mathbb{R}) \subset PT^*(\mathbb{R}^n \times \mathbb{R})$.

A submanifold $i : L \subset PT^*(\mathbb{R}^n \times \mathbb{R})$ is *a Legendrian submanifold* if $\dim L = n$ and $di_p(T_pL) \subset K_{i(p)}$ for any $p \in L$. We say that a point $p \in L$ is *a Legendrian singular point* if $\mathrm{rank}\, d(\pi \circ i)_p < n$. We also say that a point $p \in L$ is *a space-singular point* if $\mathrm{rank}\, d(\pi_1 \circ \pi \circ i)_p < n$, where $\pi_1 : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ is the canonical projection. By definition, if a point $p \in L$ is a Legendrian singular point, then it is a space-singular point. If $i : L \subset J^1(\mathbb{R}^n, \mathbb{R})$, the converse assertion also holds as the following lemma shows:

Lemma B.1. *Let $i : L \subset PT^*(\mathbb{R}^n \times \mathbb{R})$ be a Legendrian submanifold with $L \subset J^1(\mathbb{R}^n, \mathbb{R})$. Then a point $p \in L$ is a Legendrian singular point if and only if it is a space-singular point.*

Proof. Let $p \in L$ be a space-singular point. Then there exists a non-zero tangent vector $v \in T_pL$ such that $d(\pi_1 \circ \pi \circ i)_p(v) = 0$. Under the canonical coordinate of $J^1(\mathbb{R}^n, \mathbb{R})$, we have

$$i(v) = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i} + \beta \frac{\partial}{\partial t} + \sum_{j=1}^n \gamma_j \frac{\partial}{\partial p_j}$$

for some real numbers $\alpha_i, \beta, \gamma_j$. By the assumption, we have $\alpha_i = 0$ $(i = 1, \ldots, n)$. Since i is a Legendrian immersion, we have $0 = \theta(i(v)) = \beta - \sum_{i=1}^n \gamma_i \alpha_i = \beta$. It follows that

$$\mathrm{d}\pi \circ i(v) = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i} + \beta \frac{\partial}{\partial t} = 0.$$

Therefore, $p \in L$ is a Legendrian singular point. □

We also say that a point $p \in L$ is *a time-singular point* if $\mathrm{rank}\, d(\pi_2 \circ \pi \circ i)_p < 1$, where $\pi_2 : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$ is the canonical projection. Then we have the following lemma.

Lemma B.2. *Let $i : L \subset PT^*(\mathbb{R}^n \times \mathbb{R})$ be a Legendrian submanifold without Legendrian singular points. If $p \in L$ is a space-singular point, then p is not a time-singular point (i.e., $\pi_2 \circ \pi \circ i$ is a submersion at p), Moreover, under the same assumption, $i|(\pi_2 \circ \pi \circ i)^{-1}(c)$ is an $(n-1)$-dimensional isotropic immersion at p, where $c = \pi_2 \circ \pi \circ i(p)$ such that $\mathrm{rank}\, d(\pi \circ i|(\pi_2 \circ \pi \circ i)^{-1}(c))_p = n - 1$ (i.e., $\pi \circ i|(\pi_2 \circ \pi \circ i)^{-1}(c)$ is an immersion at p).*

Proof. By the assumption, $\pi \circ i$ is an immersion. For any $v \in T_pL$, there exist $X_v \in T_{\pi \circ i(p)}(\mathbb{R}^n \times \{0\})$ and $Y_v \in T_{\pi \circ i(p)}(\{0\} \times \mathbb{R})$ such that $d(\pi \circ i)_p(v) = X_v + Y_v$. If $\mathrm{rank}\, d(\pi_2 \circ \pi \circ i)_p = 0$, then $d(\pi \circ i)_p(v) = X_v$ for any $v \in T_pL$. Since p is a space-singular point, there exists a non-zero tangent vector $v \in T_pL$ such that $X_v = 0$, so that $d(\pi \circ i)_p(v) = 0$. This contradicts the fact that $\pi \circ i$ is an immersion.

Since i is a Legendrian immersion such that $\pi \circ i$ is an immersion, $i|(\pi_2 \circ \pi \circ i)^{-1}(c)$ is an $(n-1)$-dimensional isotropic immersion at p and $\pi \circ i|(\pi_2 \circ \pi \circ i)^{-1}(c)$ is also an immersion at p. □

For a Legendrian submanifold $i : L \subset PT^*(\mathbb{R}^n \times \mathbb{R})$, $\pi \circ i(L) = W(L)$ is called *a big wave front*. We have a family of *small fronts*:

$$W_t(L) = \pi_1(\pi_2^{-1}(t) \cap W(L)) \quad (t \in \mathbb{R}).$$

In this sense we call L a *big Legendrian submanifold. The discriminant* of the family $W_t(L)$ is defined as the image of singular points of $\pi_1|W(L)$. In the general case, the discriminant consists of three components: *the caustics* C_L, the projection of the set of singular points of $W(L)$, *the Maxwell stratum* M_L, the projection of self intersection points of $W(L)$; and also of the *envelope of the family of small fronts* Δ. By definition, $C_L \cup \Delta$ is the projection of space-singular points. By Lemma B.1, if $i : L \subset J^1(\mathbb{R}^n, \mathbb{R})$, then the discriminant is $C_L \cup M_L$. Moreover, by Lemma B.2, if $i : L \subset PT^*(\mathbb{R}^n \times \mathbb{R})$ is not Legendrian singular at any point, then the discriminant is $M_L \cup \Delta$.

We now consider equivalence relations among Legendrian submanifolds which preserve both of qualitative pictures of bifurcations of families of small fronts and discriminants.

Let $i : (L, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ and $i' : (L', p_0') \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0')$ be Legendrian submanifold germs. We say that i and i' are *space–time Legendrian equivalent* if there exist diffeomorphism germs $\Phi : (\mathbb{R}^n \times \mathbb{R}, \pi(p_0)) \longrightarrow (\mathbb{R}^n \times \mathbb{R}, \pi(p_0'))$ of the form $\Phi(x, t) = (\phi_1(x), \phi_2(t))$ and $\Psi : (L, p_0) \longrightarrow (L', p_0')$ such that $\widehat{\Phi} \circ i = i \circ \Psi$, where $\widehat{\Phi} : (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0) \longrightarrow (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0')$ is the unique contact diffeomorphism germ with $\pi \circ \widehat{\Phi} = \Phi \circ \pi$.

This equivalence relation is the most natural equivalence relation among Legendrian immersion germs for our purpose. It might be, however, quite hard to study because it leads the equivalence relation among divergent diagrams $\mathbb{R}^n \longleftarrow \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$. In order to avoid the difficulty, we introduce rather a strong equivalence relation as follows. We say that i and i' are *strictly parametrized Legendrian equivalent* (or briefly *$S.P$-Legendrian equivalent*) if there exist diffeomorphism germs $\Phi : (\mathbb{R}^n \times \mathbb{R}, \pi(p_0)) \longrightarrow (\mathbb{R}^n \times \mathbb{R}, \pi(p_0'))$ of the form $\Phi(x, t) = (\phi_1(x), t)$ and $\Psi : (L, p_0) \longrightarrow (L', p_0')$ such that $\widehat{\Phi} \circ i = i \circ \Psi$. Although this equivalence relation is rather easier to handle, functional modulus in the generic classification might appear even in low dimensional case. Therefore, we introduce another equivalence relation which ignore the function moduli as follows. We say that i and i' are *strictly parametrized$^+$ Legendrian equivalent* (or briefly *$S.P^+$-Legendrian equivalent*) if there exist diffeomorphism germs $\Phi : (\mathbb{R}^n \times \mathbb{R}, \pi(p_0)) \longrightarrow (\mathbb{R}^n \times \mathbb{R}, \pi(p_0'))$ of the form $\Phi(x, t) = (\phi_1(x), t + \alpha(x))$ and $\Psi : (L, p_0) \longrightarrow (L', p_0')$ such that $\widehat{\Phi} \circ i = i \circ \Psi$.

The $S.P^+$-Legendrian equivalence has been introduced in [10,11,25] for the study of completely integrable holonomic systems of first order partial differential equations. It has also been independently studied by Zakalyukin [28] called *the strongly space-equivalence*. We remark that the above equivalence relation among big Legendrian submanifold germs preserve both the diffeomorphism types of bifurcations for families of small fronts and caustics.

We study the $S.P^+$-Legendrian equivalence by using the notion of generating families of Legendrian submanifold germs.

For any Legendrian submanifold germ $i : (L, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$, there exists a generating family of i by the Arnol'd–Zakalyukin's theory [1]. Let $F : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \longrightarrow (\mathbb{R}, 0)$ be a function germ such that $(F, d_2F) : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}, 0) \longrightarrow (\mathbb{R} \times \mathbb{R}^k, 0)$ is non-singular, where

$$d_2F(q, x, t) = \left(\frac{\partial F}{\partial q_1}(q, x, t), \ldots, \frac{\partial F}{\partial q_k}(q, x, t)\right).$$

In this case we call F a *big Morse family of hypersurfaces*. Then $\Sigma_*(F) = (F, d_2F)^{-1}(0)$ is a smooth n-manifold germ. Define

$$\mathcal{L}_F : (\Sigma_*(F), 0) \longrightarrow PT^*(\mathbb{R}^n \times \mathbb{R})$$

by

$$\mathcal{L}_F(q, x, t) = \left(x, t, \left[\frac{\partial F}{\partial x}(q, x, t) : \frac{\partial F}{\partial t}(q, x, t)\right]\right),$$

where

$$\left[\frac{\partial F}{\partial x}(q,x,t) : \frac{\partial F}{\partial t}(q,x,t)\right] = \left[\frac{\partial F}{\partial x_1}(q,x,t) : \cdots : \frac{\partial F}{\partial x_n}(q,x,t) : \frac{\partial F}{\partial t}(q,x,t)\right].$$

It is easy to show that $\mathcal{L}_F(\Sigma_*(F))$ is a Legendrian submanifold germ. By the main theorem of Arnol'd–Zakalyukin [1], we can show the following proposition:

Proposition B.3. *All big Legendrian submanifold germs are constructed by the above method.*

Let $F : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \longrightarrow (\mathbb{R}, 0)$ be a big Morse family of hypersurfaces. We call F a *generating family of* $\mathcal{L}_F$. We now consider ambiguity of the choice of generating families. Let $F, G : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \longrightarrow (\mathbb{R}, 0)$ be big Morse families. We say that F and G are *strictly $\mathcal{R}$-equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \longrightarrow (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0)$ of the form $\Phi(q,x,t) = (\phi(q,x,t), x, t)$ such that $F \circ \Phi = G$. If we carefully read proofs of Lemmas 1 and 2 in ([1], page 307), we can understand the following assertion.

Proposition B.4. *Let $F, G : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \longrightarrow (\mathbb{R}, 0)$ be big Morse families of hypersurfaces such that*

$$\operatorname{Image}\mathcal{L}_F = \operatorname{Image}\mathcal{L}_G \quad \textit{and} \quad \operatorname{rank}\mathcal{H}(F|\mathbb{R}^k \times \{0\})(0) = \operatorname{rank}\mathcal{H}(G|\mathbb{R}^k \times \{0\})(0) = 0,$$

where $\mathcal{H}(f)$ is the Hessian matrix of f. Then F and G are strictly $\mathcal{R}$-equivalent.

Let $f, g : (\mathbb{R}^k \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ be function germs. We say that f and g are *S.P-$\mathcal{K}$-equivalent* (or *strictly P-$\mathcal{K}$-equivalent*) if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^k \times \mathbb{R}, 0) \longrightarrow (\mathbb{R} \times \mathbb{R}^k, 0)$ of the form $\Phi(q,t) = (\phi(q,t), t)$ such that $\langle f \circ \Phi\rangle_{\mathcal{E}_{(q,t)}} = \langle g\rangle_{\mathcal{E}_{(q,t)}}$.

Let $F, G : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \longrightarrow (\mathbb{R}, 0)$ be function germs. We say that F and G are *x-$S.P^+$-$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \longrightarrow (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0)$ of the form $\Phi(q,x,t) = (\phi(q,x,t), \phi_2(x), t + \alpha(x))$ such that $\langle F \circ \Phi\rangle_{\mathcal{E}_{(q,x,t)}} = \langle G\rangle_{\mathcal{E}_{(q,x,t)}}$, where $x = (x_1, \ldots, x_n)$ is the canonical coordinate of $(\mathbb{R}^n, 0)$.

The notion of $S.P^+$-$\mathcal{K}$-versal deformation plays an important role for our purpose. We define the extended tangent space of $f : (\mathbb{R}^k \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ relative to $S.P^+$-$\mathcal{K}$ by

$$T_e(S.P^+\text{-}\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \ldots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\mathcal{E}_{(q,t)}} + \left\langle \frac{\partial f}{\partial t} \right\rangle_{\mathbb{R}}.$$

Then we say that a deformation F of $f = F|_{\mathbb{R}^k \times \{0\} \times \mathbb{R}}$ is *infinitesimally $S.P^+$-$\mathcal{K}$-versal* if it satisfies

$$\mathcal{E}_{(q,t)} = T_e(S.P^+\text{-}\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1}\middle| \mathbb{R}^k \times \{0\} \times \mathbb{R}, \ldots, \frac{\partial F}{\partial x_n}\middle| \mathbb{R}^k \times \{0\} \times \mathbb{R} \right\rangle_{\mathbb{R}}.$$

We simply say that F is a *$S.P^+$-$\mathcal{K}$-versal deformation* of f if it is infinitesimally $S.P^+$-$\mathcal{K}$-versal.

We remark that F is $S.P^+$-$\mathcal{K}$-versal, then n is upper bound for

$$\dim_{\mathbb{R}} \mathcal{E}_{(q,t)}/T_e(S.P^+\text{-}\mathcal{K})(f).$$

Moreover, we have the following very important property as a consequence of the versality theorem [5].

Proposition B.5. (1) *Suppose that F, G be n-parameter $S.P^+$-$\mathcal{K}$-versal deformations of f. Then F and G are x-$S.P^+$-$\mathcal{K}$-equivalent.*

(2) *Let $\xi_1(q,t), \ldots, \xi_n(q,t)$ be generators of the $\mathbb{R}$-vector space $\mathcal{E}_{(q,t)}/T_e(S.P^+\text{-}\mathcal{K})(f)$, then any n-parameter $S.P^+$-$\mathcal{K}$-versal deformations are x-$S.P^+$-$\mathcal{K}$-equivalent to*

$$F(q,x,t) = f(q,t) + \sum_{i=1}^{n} x_i \xi_i(q,t).$$

Theorem B.6. *Let $F : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \longrightarrow (\mathbb{R}, 0)$ and $G : (\mathbb{R}^{k'} \times (\mathbb{R}^n \times \mathbb{R}), 0) \longrightarrow (\mathbb{R}, 0)$ be big Morse families of hypersurfaces. Then*

(1) *$\mathcal{L}_F$ and $\mathcal{L}_G$ are $S.P^+$-Legendrian equivalent if and only if F and G are stably x-$S.P^+$-$\mathcal{K}$-equivalent.*
(2) *$\mathcal{L}_F$ is a $S.P^+$-Legendrian stable if and only if F is a $S.P^+$-$\mathcal{K}$-versal deformation of $f = F|\mathbb{R}^k \times \{0\} \times \mathbb{R}$.*

Here, F and G are said to be stably x-$S.P^+$-$\mathcal{K}$-equivalent if they become x-$S.P^+$-$\mathcal{K}$-equivalent after the addition of non-degenerate quadratic forms in additional variables q'.

We have another characterization of the $S.P^+$-$\mathcal{K}$-versality for families of function germs. For any function germ $F : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \longrightarrow (\mathbb{R}, 0)$, we have the r-jet extension

$$j_1^r F : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \longrightarrow J^r(\mathbb{R}^k \times \mathbb{R}, \mathbb{R})$$

defined by $j_1^r F(q, x, t) = j^r F_x(q, t)$, where $F_x(q, t) = F(q, x, t)$. On the other hand, we have $(S.P\text{-}\mathcal{K})^r$-orbits in $J^r(k + 1, 1)$, where we have the canonical decomposition $J^r(\mathbb{R}^k \times \mathbb{R}, \mathbb{R}) = (\mathbb{R}^k \times \mathbb{R}) \times \mathbb{R} \times J^r(k + 1, 1)$. For any $z = j^r f(0) \in J^r(k + 1, 1)$, we define that

$$\widetilde{(S.P\text{-}\mathcal{K})^r}(z) = (\mathbb{R}^k \times \mathbb{R}) \times \{0\} \times (S.P\text{-}\mathcal{K})^r(z),$$

where $(S.P\text{-}\mathcal{K})^r(z)$ is the $(S.P\text{-}\mathcal{K})^r$-orbit through z.

Proposition B.7. *Suppose that $f = F|\mathbb{R}^k \times \{0\} \times \mathbb{R}$ is r-determined relative to $S.P$-$\mathcal{K}$ (for the definition, see [8]). The following conditions are equivalent:*

(1) *F is a $S.P^+$-$\mathcal{K}$-versal deformation of f.*
(2) *$j_1^r F$ is transverse to $\widetilde{(S.P\text{-}\mathcal{K})^r}(z)$, where $z = j^r f(0)$.*

Since the big Legendrian submanifold germ $i : (L, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ is uniquely determined on the regular part of the big wave front $W(L)$, we have the following simple but significant property of Legendrian submanifold germs:

Proposition B.8. *Let $i : (L, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ and $i' : (L', p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ be big Legendrian submanifold germs such that regular sets of $\pi \circ i, \pi \circ i'$ are dense respectively. Then $(L, p_0) = (L', p_0)$ if and only if $(W(L), \pi(p_0)) = (W(L'), \pi(p_0))$.*

This result has been firstly pointed out by Zakalyukin [27]. The assumption in the above proposition is a generic condition for i, i'. Specially, if i and i' are $S.P^+$-Legendrian stable, then these satisfy the assumption. Concerning the discriminant and the bifurcation of small fronts, we define the following equivalence relation among big wave front germs. Let $i : (L, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ and $i' : (L', p_0') \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0')$ be Legendrian submanifold germs. We say that $W(L)$ and $W(L')$ are *$S.P^+$-diffeomorphic* if there exists diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}, \pi(p_0)) \longrightarrow (\mathbb{R}^n \times \mathbb{R}, \pi(p_0'))$ of the form $\Phi(x, t) = (\phi_1(x), t + \alpha(x))$ such that $\Phi(W(L)) = W(L')$. By Proposition B.8, we have the following proposition.

Proposition B.9. *Let $i : (L, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ and $i' : (L', p_0') \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0')$ be big Legendrian submanifold germs such that regular sets of $\pi \circ i, \pi \circ i'$ are dense respectively. Then i and i' are $S.P^+$-Legendrian equivalent if and only if $(W(L), \pi(p_0))$ and $(W(L'), \pi(p_0'))$ are $S.P^+$-diffeomorphic.*

Appendix C. Graphlike Legendrian unfoldings

In this appendix, we consider a special class of Legendrian submanifolds in $J^1(\mathbb{R}^n, \mathbb{R}) \subset PT^*(\mathbb{R}^n \times \mathbb{R})$. We say that a Legendrian submanifold $i : L \subset J^1(\mathbb{R}^n, \mathbb{R})$ is a *graphlike Legendrian unfolding* if $\pi_2 \circ \pi \circ i$ is a submersion (i.e., time-nonsingular) at any point $p \in L$. The notion of graphlike Legendrian unfoldings has been introduced by the first named author [9] in order to describe the perestroikas of wave fronts given as the level surfaces of the solution for the eikonal equation given by a general Hamiltonian function. Since L is a Legendrian submanifold in $J^1(\mathbb{R}^n, \mathbb{R})$, it has a big generating family at least locally. In this case it has a special form as follows. Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \longrightarrow (\mathbb{R}, 0)$ be a big Morse family of hypersurfaces. We say that $\mathcal{F}$ is a *graphlike Morse family of hypersurfaces* if $(\partial \mathcal{F}/\partial t)(0) \neq 0$. It is easy to show that the corresponding big Legendrian submanifold germ is a graphlike Legendrian unfolding. Of course all graphlike Legendrian unfolding germs can be constructed

by the above way. In this case we say that $\mathcal{F}$ is a *graphlike generating family* of $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$. However, we can reduce the more strict form of graphlike generating families. Let $\mathcal{F}$ be a graphlike Morse family of hypersurfaces. By the implicit function theorem, there exists a Morse family of functions $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ such that $\langle \mathcal{F}(q, x, t)\rangle_{\mathcal{E}(q,x,t)} = \langle F(q, x) - t\rangle_{\mathcal{E}(q,x,t)}$. Therefore $F(q, x) - t$ is a graphlike generating family of $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$. In this case

$$\Sigma_*(F) = \{(q, x, F(q, x)) \in (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \mid (q, x) \in C(F)\}$$

and

$$\mathcal{L}_{\mathcal{F}}(q, x, F(q, x)) = (L(F)(q, x), F(q, x)) \in J^1(\mathbb{R}^n, \mathbb{R}) \equiv T^*\mathbb{R}^n \times \mathbb{R}.$$

Define a map $\mathfrak{L}_F : C(F) \longrightarrow J^1(\mathbb{R}^n, \mathbb{R})$ by $\mathfrak{L}_F(q, x) = (q, x, F(q, x), (\partial F/\partial x)(q, x))$, then we have $\mathfrak{L}_F(C(F)) = \mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$. We call $W(\mathfrak{L}_F) = \pi(\mathfrak{L}_F(C(F)))$ the *wave fronts* of graphlike Legendrian unfolding $\mathfrak{L}_F$. We simply call F a *generating family* of the graphlike Legendrian unfolding $\mathfrak{L}_F$. For any Morse family of function F, we denote that $\overline{F}(q, x, t) = F(q, x) - t$. Since $\overline{F}(q, x, t)$ is a big Morse family, we can use all the definitions of equivalence relations given in Appendix B. Moreover we can translate the propositions and theorems into corresponding assertions in terms of graphlike Legendrian unfoldings. We denote that $\overline{f}(q, t) = f(q) - t$ for any $f \in \mathfrak{M}_k$. Then we can represent the extended tangent space of $\overline{f} : (\mathbb{R}^k \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ relative to $S.P^+$-$\mathcal{K}$ by

$$T_e(S.P^+\text{-}\mathcal{K})(\overline{f}) = \left\langle \frac{\partial f}{\partial q_1}(q), \ldots, \frac{\partial f}{\partial q_k}(q), f(q) - t \right\rangle_{\mathcal{E}_{(q,t)}} + \langle 1\rangle_{\mathbb{R}}.$$

For a deformation $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ of f, $\overline{F}$ is $S.P^+$-$\mathcal{K}$-versal deformation of $\overline{f}$ if and only if

$$\mathcal{E}_{(q,t)} = T_e(S.P^+\text{-}\mathcal{K})(\overline{f}) + \left\langle \frac{\partial F}{\partial x_1}\middle| \mathbb{R}^k \times \{0\}, \ldots, \frac{\partial F}{\partial x_n}\middle| \mathbb{R}^k \times \{0\} \right\rangle_{\mathbb{R}}.$$

Moreover, we have the following very important property as a consequence of the versality theorem [5].

Theorem C.1. *Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ and $G : (\mathbb{R}^{k'} \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ be Morse families of functions. Then*

(1) *$\mathfrak{L}_F$ and $\mathfrak{L}_G$ are $S.P^+$-Legendrian equivalent if and only if $\overline{F}$ and $\overline{G}$ are stably x-$S.P^+$-$\mathcal{K}$-equivalent.*
(2) *$\mathfrak{L}_F$ is $S.P^+$-Legendrian stable if and only if $\overline{F}$ is an $S.P^+$-$\mathcal{K}$-versal deformation of $f = F|\mathbb{R}^k \times \{0\}$.*

By Proposition A.1, any Lagrangian submanifold germ in $T^*\mathbb{R}^n$ is given by $L(F)(C(F))$ for a Morse family of functions F. Let F, G be Morse families of functions, then $L(F)(C(F))$ and $L(G)(C(G))$ are Lagrangian equivalent if and only if F and G are stably P-$\mathcal{R}^+$-equivalent (cf. Theorem A.2). By definition, if F and G are stably P-$\mathcal{R}^+$-equivalent, then $\overline{F}$ and $\overline{G}$ are stably x-$S.P^+$-$\mathcal{K}$-equivalent. Therefore we have the following proposition.

Proposition C.2. *If $L(F)(C(F))$ and $L(G)(C(G))$ are Lagrangian equivalent, then $\mathfrak{L}_F(C(F))$ and $\mathfrak{L}_G(C(G))$ are $S.P^+$-Legendrian equivalent.*

Remark. The above proposition asserts that the Lagrangian equivalence is stronger equivalence relation than the $S.P^+$-Legendrian equivalence. The $S.P^+$-Legendrian equivalence relation among graphlike Legendrian unfoldings preserves both the diffeomorphism types of bifurcations for families of small fronts and caustics. On the other hand, if we observe the real caustics of rays, we cannot observe the structure of wave front propagations. In this sense, there are hidden structures behind the picture of real caustics (cf. Appendix B). By the above proposition, the Lagrangian equivalence preserve not only the diffeomorphism type of caustics but also the hidden geometric structure of wave front propagations.

Moreover, suppose that $\overline{F}$ is a $S.P^+$-$\mathcal{K}$-versal deformation of $\overline{f}$. By definition, we have

$$\mathcal{E}_{(q,t)} = T_e(S.P^+\text{-}\mathcal{K})(\overline{f}) + \left\langle \frac{\partial F}{\partial x_1}\middle| \mathbb{R}^k \times \{0\}, \ldots, \frac{\partial F}{\partial x_n}\middle| \mathbb{R}^k \times \{0\} \right\rangle_{\mathbb{R}}.$$

The proof of the following proposition is given in [19].

Proposition C.3 (*[19, Proposition 4.1]*). *Let* $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ *be a Morse family of functions. If* $\mathfrak{L}_F$ *is* $S.P^+$*-Legendrian stable, then* $L(F)$ *is Lagrangian stable.*

References

[1] V.I. Arnol'd, S.M. Gusein-Zade, A.N. Varchenko, Singularities of Differentiable Maps, vol. I, Birkhäuser, 1986.
[2] V.I. Arnol'd, Singularities of Caustics and Wave Fronts, in: Mathematics and its Applications, vol. 62, Kluwer, 1990.
[3] A.C. Asperti, M. Dajczer, Conformally flat Riemannian manifolds as hypersurfaces of the lightcone, Canad. Math. Bull. 32 (1989) 281–285.
[4] J.W. Bruce, Wavefronts and parallels in Euclidean space, Math. Proc. Cambridge Philos. Soc. 93 (1983) 323–333.
[5] J. Damon, The unfolding and determinacy theorems for subgroups of $\mathcal{A}$ and $\mathcal{K}$, Mem. Amer. Math. Soc. 50 (306) (1984).
[6] M. Golubitsky, V. Guillemin, Contact equivalence for Lagrangian manifold, Adv. Math. 15 (1975) 375–387.
[7] V.V. Goryunov, Projections of generic surfaces with boundaries, Adv. Soviet Math. 1 (1990) 157–200.
[8] S. Izumiya, Generic bifurcations of varieties, Manuscripta Math. 46 (1984) 137–164.
[9] S. Izumiya, Perestroikas of optical wave fronts and graphlike Legendrian unfoldings, J. Differential Geom. 38 (1993) 485–500.
[10] S. Izumiya, Completely integrable holonomic systems of first-order differential equations, Proc. Roy. Soc. Edinburgh 125A (1995) 567–586.
[11] S. Izumiya, Y. Kurokawa, Holonomic systems of Clairaut type, Differential Geom. Appl. 5 (1995) 219–235.
[12] S. Izumiya, D.-H. Pei, T. Sano, Singularities of hyperbolic Gauss maps, Proc. London Math. Soc. 86 (2003) 485–512.
[13] S. Izumiya, D.-H. Pei, T. Sano, E. Torii, Evolutes of hyperbolic plane curves, Acta Math. Sinica 20 (2004) 543–550.
[14] S. Izumiya, D.-H. Pei, M. Takahashi, Singularities of evolutes of hypersurfaces in hyperbolic space, Proc. Edinb. Math. Soc. 47 (2004) 131–153.
[15] S. Izumiya, D.-H. Pei, M.C. Romero Fuster, M. Takahashi, On the horospherical ridges of submanifolds of codimension 2 in hyperbolic n-space, Bull. Braz. Math. Soc. 35 (2004) 177–198.
[16] S. Izumiya, M.C. Romero Fuster, D.-H. Pei, M. Takahashi, The horospherical geometry of submanifolds in hyperbolic n-space, J. London Math. Soc. 71 (2005) 779–800.
[17] S. Izumiya, D.-H. Pei, M.C. Romero Fuster, The horospherical geometry of submanifolds in hyperbolic 4-space, Israel J. Math. 154 (2006) 361–379.
[18] S. Izumiya, Legendrian dualities and spacelike hypersurfaces in the lightcone, Preprint.
[19] S. Izumiya, M. Takahashi, Caustics and wave front propagations: Applications to differential geometry, Preprint (2006).
[20] J. Martinet, Singularities of Smooth Functions and Maps, in: London Math. Soc. Lecture Note Series, vol. 58, Cambridge Univ. Press, 1982.
[21] J.A. Montaldi, On contact between submanifolds, Michigan Math. J. 33 (1986) 195–199.
[22] J.A. Montaldi, On generic composites of maps, Bull. London Math. Soc. 23 (1991) 81–85.
[23] B. O'neil, Semi-Riemannian Geometry, Academic Press, New York, 1983.
[24] I. Porteous, The normal singularities of submanifold, J. Differential Geom. 5 (1971) 543–564.
[25] M. Takahashi, Holonomic systems of general Clairaut type, Hokkaido Math. J. 34 (2005) 247–263.
[26] G. Wassermann, Stability of caustics, Math. Ann. 216 (1975) 43–50.
[27] V.M. Zakalyukin, Reconstructions of fronts and caustics depending on a parameter and versality of mappings, J. Soviet Math. 27 (1983) 2713–2735.
[28] V.M. Zakalyukin, Envelope of families of wave fronts and control theory, Proc. Steklov Inst. Math. 209 (1995) 114–123.